

Global smooth solutions of 3-D quasilinear wave equations with small initial data

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Abstract

In this paper, we are concerned with the 3-D quasilinear wave equation $\sum_{i,j=0}^3 g^{ij}(u, \partial u) \partial_{ij}^2 u = 0$ with $(u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$, where $x_0 = t$, $x = (x_1, x_2, x_3)$, $\partial = (\partial_0, \partial_1, \dots, \partial_3)$, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^3)$, $\varepsilon > 0$ is small enough, and $g^{ij}(u, \partial u) = g^{ji}(u, \partial u)$ are smooth in their arguments. Without loss of generality, one can write $g^{ij}(u, \partial u) = c^{ij} + d^{ij}u + \sum_{k=0}^3 e_k^{ij} \partial_k u + O(|u|^2 + |\partial u|^2)$, where c^{ij} , d^{ij} and e_k^{ij} are some constants, and $\sum_{i,j=0}^3 c^{ij} \partial_{ij}^2 = -\square \equiv -\partial_t^2 + \Delta$. When $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \not\equiv 0$ for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$, the authors in [7-8] have shown the blowup of the smooth solution u in finite time as long as $(u_0(x), u_1(x)) \not\equiv 0$. In the present paper, when $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$, we will prove the global existence of the smooth solution u . Therefore, the complete results on the blowup or global existence of the small data solutions have been established for the general 3-D quasilinear wave equations $\sum_{i,j=0}^3 g^{ij}(u, \partial u) \partial_{ij}^2 u = 0$.

Keywords: Global existence, null frame, null condition, weighted energy estimate, Poincaré inequality

Mathematical Subject Classification 2000: 35L05, 35L72

§1. Introduction and main results

*This project was supported by the NSFC (No. 11025105), and by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

Consider the second order quasilinear wave equation in $[0, \infty) \times \mathbb{R}^n$

$$\begin{cases} \tilde{\square}_g u \equiv \sum_{i,j=0}^n g^{ij}(u, \partial u) \partial_{ij}^2 u = 0, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \quad (1.1)$$

where $x_0 = t$, $x = (x_1, \dots, x_n)$, $\partial = (\partial_0, \partial_1, \dots, \partial_n)$, $\varepsilon > 0$ is a sufficiently small constant, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^n)$, $g^{ij}(u, \partial u) = g^{ji}(u, \partial u)$ are smooth functions which can be certainly expressed as

$$g^{ij}(u, \partial u) = c^{ij} + d^{ij}u + \sum_{k=0}^n e_k^{ij} \partial_k u + O(|u|^2 + |\partial u|^2) \quad (1.2)$$

with c^{ij} , d^{ij} and e_k^{ij} being some constants. Without loss of generality, one can assume $\sum_{i,j=0}^n c^{ij} \partial_{ij}^2 = -\square \equiv -\partial_t^2 + \Delta$. By the well-known results in [11], [15-18] and references therein, we know that (1.1) has a global smooth solution for $n \geq 4$.

If $n = 3$ and $d^{ij} = 0$ for all $0 \leq i, j \leq 3$ in (1.2), then (1.1) has a global smooth solution if the null condition holds (namely, $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$ holds for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$), otherwise, the solution of (1.1) will blow up in finite time. See [3], [6], [10-14], [21-23] and the references therein.

If $n = 3$ and $d^{ij} \neq 0$ for some (i, j) , but $e_k^{ij} = 0$ for all $0 \leq i, j, k \leq 3$ in (1.2), then it follows from the results in [4] and [19-20] that (1.1) has a global smooth solution.

If $n = 3$, $d^{ij} \neq 0$ for some (i, j) and $e_k^{ij} \neq 0$ for some (i, j, k) in (1.2), when $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \neq 0$ for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$, we have established the following blowup result (see Remark 1.2 in [7] and Remark 1.4 in [8]): Assume $(u_0(x), u_1(x)) \neq 0$ and denote by T_ε the lifespan of the smooth solution u to (1.1), then there exists a positive constant τ_0 depending only on $(u_0(x), u_1(x))$ and the coefficients d^{ij} , e_k^{ij} in (1.2) such that $\lim_{\varepsilon \rightarrow 0^+} \varepsilon \ln T_\varepsilon = \tau_0$.

If $n = 3$, $d^{ij} \neq 0$ for some (i, j) , and $e_k^{ij} \neq 0$ for some (i, j, k) in (1.2), when $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$, a basic problem naturally arises: does the smooth solution of (1.1) blow up in finite time or exist globally? We will study this problem for the equation (1.1) in the case of $n = 3$

$$\begin{cases} \tilde{\square}_g u \equiv \sum_{i,j=0}^3 g^{ij}(u, \partial u) \partial_{ij}^2 u = 0, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \quad (1.3)$$

where $u_0(x), u_1(x) \in C_0^\infty(B(0, 1))$, and $B(0, 1)$ represents a unit ball centered at the origin O . Since the higher order terms $O(|u|^2) \partial_{ij}^2 u$ and $O(|\partial u|^2) \partial_{ij}^2 u$ in (1.3) do not play the essential

roles in our study, without loss of generality, we assume that the smooth coefficients

$$g^{ij}(u, \partial u) = g^{ji}(u, \partial u) = c^{ij} + d^{ij}u + \sum_{k=0}^3 e_k^{ij} \partial_k u, \quad i, j = 0, 1, 2, 3. \quad (1.4)$$

In addition, $\sum_{i,j=0}^3 c^{ij} \partial_{ij}^2 = -\square$ is still assumed. The main conclusion in this paper is:

Theorem 1.1. *Under the above assumptions on the problem (1.3), if $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$ holds, then there exists an $\varepsilon_0 > 0$ such that the problem (1.3) has a global C^∞ solution $u(t, x)$ for $\varepsilon < \varepsilon_0$.*

Remark 1.1. *The detailed decay properties of the solution $u(t, x)$ to (1.3) will be given in (3.57)-(3.61) of Proposition 3.9 in §3 below. From Proposition 3.9, as in [4] and [19-20], we know that the solution of (1.3) does not decay like the solution of the 3-D free wave equation $\square v = 0$ with $(v(0, x), \partial_t v(0, x)) = (u_0(x), u_1(x))$ since $|v| \leq C(1+t)^{-1}(1+|r-t|)^{-\frac{1}{2}}$ holds.*

Remark 1.2. *For the 2-D nonlinear wave equation whose coefficients depend only on ∂u*

$$\begin{cases} \sum_{i,j=0}^2 g^{ij}(\partial u) \partial_{ij}^2 u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^2, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \quad (1.5)$$

where $g^{ij}(\partial u) = g^{ji}(\partial u) = c^{ij} + \sum_{k=0}^2 e_k^{ij} \partial_k u + \sum_{k,l=0}^2 e_{kl}^{ij} \partial_k u \partial_l u + O(|\partial u|^3)$, $\varepsilon > 0$ is sufficiently

small, and $\sum_{i,j=0}^2 c^{ij} \partial_{ij}^2 u = -\square$. As $\sum_{i,j,k=0}^2 e_k^{ij} \omega_k \omega_i \omega_j \not\equiv 0$ or $\sum_{i,j,k,l=0}^2 e_{kl}^{ij} \omega_k \omega_l \omega_i \omega_j \not\equiv 0$ for $\omega_0 = -1$

and $\omega = (\omega_1, \omega_2) \in \mathbb{S}^1$, it is well-known that the smooth solution to (1.5) must blow up in finite

time as long as $(u_0(x), u_1(x)) \not\equiv 0$ (see [1], [3], [9-11], [18] and so on). As $\sum_{i,j,k=0}^2 e_k^{ij} \omega_k \omega_i \omega_j \equiv$

0 and $\sum_{i,j,k,l=0}^2 e_{kl}^{ij} \omega_k \omega_l \omega_i \omega_j \equiv 0$, (1.5) has a global smooth solution (see [2]). In summary, the

complete results on the blowup or global existence have been established for the small data smooth solution problem (1.5).

Remark 1.3. *Consider more general 2-D nonlinear wave equations whose coefficients depend on the solution u as well as its first order derivatives ∂u*

$$\begin{cases} \sum_{i,j=0}^2 g^{ij}(u, \partial u) \partial_{ij}^2 u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^2, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \quad (1.6)$$

where $g^{ij}(u, \partial u) = g^{ji}(u, \partial u) = c^{ij} + d^{ij}u + \sum_{k=0}^2 e_k^{ij} \partial_k u + \tilde{d}^{ij}u^2 + \sum_{k=0}^2 \tilde{e}_k^{ij} u \partial_k u + \sum_{k=0}^2 e_{kl}^{ij} \partial_k u \partial_l u + O(|u|^3 + |\partial u|^3)$, $d^{ij} \neq 0$ for some (i, j) , $\varepsilon > 0$ is sufficiently small, and $\sum_{i,j=0}^2 c^{ij} \partial_{ij}^2 u = -\square$. When

$\sum_{i,j,k=0}^2 e_k^{ij} \omega_k \omega_i \omega_j \not\equiv 0$ and $(u_0(x), u_1(x)) \not\equiv 0$, we have shown the blowup of the smooth solution u to (1.6) in finite time and further given a precise description of the blowup mechanism in [7]. With respect to the other left cases for the coefficients d^{ij} , e_k^{ij} , \tilde{d}^{ij} , \tilde{e}_k^{ij} and e_{kl}^{ij} of $g^{ij}(u, \partial u)$, by our knowledge so far there are no complete results on the blowup or global existence of the solution u to (1.6).

We now give the comments on the proof of Theorem 1.1. To prove Theorem 1.1, we will use the usual energy method and the continuity argument. For this purpose, we define the standard energy $E_N(t) = \sum_{0 \leq I \leq N} \int |\partial Z^I u|^2 dx$ of (1.3) for some suitably large integer N , where

Z denotes one of the Klainerman's fields $\{\partial_t, \partial_\alpha, S = t\partial_t + \sum_{i=1}^3 x_i \partial_i, \Gamma_{0\alpha} = t\partial_\alpha + x_\alpha \partial_t, \Gamma_{\alpha\beta} = x_\beta \partial_\alpha - x_\alpha \partial_\beta, \alpha, \beta = 1, 2, 3\}$. Motivated by [4] and [19-20], where the global existence of small data solutions to the 3-D quasilinear wave equations $\sum_{i,j=0}^3 g^{ij}(u) \partial_{ij}^2 u = 0$ is established, we think that the solution of (1.3) does not decay like the solution of the free linear wave equation

even if the null condition (i.e., $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$ for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$) holds. To prove the global existence of the solution to (1.3), as in [20], at first we assume that

$E_N(t) \leq C\varepsilon^2(1+t)^\delta$ holds for $t \in [0, T]$ and some fixed constant δ with $0 < \delta < \frac{1}{2}$, then we manage to derive a strong energy estimate $E_N(t) \leq C\varepsilon^2(1+t)^{C\varepsilon}$ for sufficiently small $\varepsilon > 0$. In this process, we will adopt the energy method with some special weight depending on the solution of an approximate eikonal equation of (1.3) (see (2.34) in §2 below) as well as the

property of the null condition $\sum_{i,j,k=0}^3 e_k^{ij} \omega_k \omega_i \omega_j \equiv 0$. Here we point out that although some main

procedures in this paper are analogous to those in [20] for considering the 3-D wave equations $\sum_{i,j=0}^3 g^{ij}(u) \partial_{ij}^2 u = 0$, our analysis is more involved since the coefficients of (1.3) depend on u and ∂u simultaneously, and meanwhile the null condition property of (1.3) should be specially paid attention. Finally, based on the apriori energy estimates mentioned above, we can complete the proof of Theorem 1.1 by the local existence of the solution to (1.3) and the continuity argument.

The rest of the paper is organized as follows. In §2, we will give some preliminary knowledge on the null frame in Lorentzian metric $g^{ij}(u, \partial u)$ and some properties of null conditions,

where the null frames in Lorentzian metric $g^{ij}(u)$ are introduced and applied in [5] and [20]. In addition, some useful estimates and calculations are listed or proved. In §3, we will derive the sharp decay estimate of the solution u to (1.3) under the assumption of the weak decay estimate $|Z^I u| \leq C\varepsilon(1+t)^{-\nu}$ for $1/2 < \nu < 1$ and $0 \leq I \leq N-3$. In §4, under some suitable assumptions on the solution u to (1.3), the global weighted energy estimate on u is established by choosing an appropriate weight so that the null condition can be utilized sufficiently. In §5, by establishing a Poincaré-type lemma similar to Lemma 8.1 of [20], we can obtain the higher order energy estimate on the solution u to (1.3). Finally, in §6, we will derive the precise energy estimate $E_N(t) \leq C\varepsilon^2(1+t)^{C\varepsilon}$. Meanwhile, all the assumptions on u in §3-§5 are closed. Therefore, the proof of Theorem 1.1 is completed by continuity argument.

In the whole paper, we will use the following notations:

Let Z denote one of the Klainerman's fields

$$\partial_t, \partial_\alpha, S = t\partial_t + \sum_{i=1}^3 x_i \partial_i, \Gamma_{0\alpha} = t\partial_\alpha + x_\alpha \partial_t, \Gamma_{\alpha\beta} = x_\beta \partial_\alpha - x_\alpha \partial_\beta, \alpha, \beta = 1, 2, 3.$$

Let ∂ stand for ∂_t or ∂_α ($\alpha = 1, 2, 3$). The norm $\|f\|_{L^2}$ means $\|f(t, \cdot)\|_{L^2(\mathbb{R}^3)}$. C denotes by a generic positive constant, and $|Z^k v| \equiv \sum_{|\nu|=k} |Z^\nu v|$ for $k \in \mathbb{N} \cup \{0\}$ (or bold \mathbf{k} , $I, J, K \in \mathbb{N} \cup \{0\}$) and the multiple indices ν 's. Specially, we denote $|Zu|$ by $|Z^1 u|$.

§2. Preliminaries

As in [5] and [20], we introduce the following nullframe $\{L, \underline{L}, S_1, S_2\}$ for the Minkowski metric $ds^2 = -dx_0^2 + \sum_{\alpha=1}^3 dx_\alpha^2$:

$$\begin{cases} L = L^i \partial_i, & \underline{L} = \underline{L}^i \partial_i, \\ S_1 = \frac{1}{\sqrt{1 - (\omega^3)^2}} (\omega^3 \omega^\alpha - \delta^{\alpha 3}) \partial_\alpha, & S_2 = \frac{1}{\sqrt{1 - (\omega^3)^2}} (-\omega^2 \partial_1 + \omega^1 \partial_2), \end{cases} \quad (2.1)$$

here and in what follows the repeated upper and lower indices stand for the summations over $i = 0, 1, 2, 3$ (or $j, k, m = 0, 1, 2, 3$) and $\alpha = 1, 2, 3$ (or $\beta = 1, 2, 3$) respectively, and

$$L^0 = \underline{L}^0 = 1, \quad L^\alpha = \omega^\alpha, \quad \underline{L}^\alpha = -\omega^\alpha, \quad \omega^\alpha = \frac{x_\alpha}{|x|}, \quad \alpha = 1, 2, 3.$$

In addition, one can raise and lower the indices with respect to the Minkowski metric $ds^2 = -dx_0^2 + \sum_{\alpha=1}^3 dx_\alpha^2 \equiv \sum_{i,j=0}^3 c_{ij} dx_i dx_j$ as follows: $X_i = c_{ij} X^j$, $X^j = c^{ij} X_j$, where the matrix $(c^{ij})_{i,j=0}^3$ stands for the inverse of the matrix $(c_{ij})_{i,j=0}^3$ (more concretely, $c^{00} = -1$, $c^{\alpha\alpha} = 1$, and $c^{ij} = 0$ for $i \neq j$, which coincides with the assumption of $c^{ij} \partial_{ij}^2 = -\square$ for the equation (1.3)).

Since $\{L, \underline{L}, S_1, S_2\}$ is a null frame, we can make a decomposition for the Lorentian metric g^{ij} in the equation (1.3) (here one notes that $g^{ij}(u, \partial u)$ in (1.3) is somewhat different from $g^{ij}(u)$ in [20]):

$$g^{ij} = g^{UV} U^i V^j, \quad (2.2)$$

where $U, V \in \{L, \underline{L}, S_1, S_2\}$. As in (2.7) of [20], we define

$$g_{UV} = g^{ij} U_i V_j, \quad (2.3)$$

which denotes the lowering of the indices and not the inverse of g^{UV} . By (2.6) of [20], g^{UV} and g_{UV} have the following relations:

$$g^{L\underline{L}} = \frac{1}{4} g_{\underline{L}L}, \quad g^{LL} = \frac{1}{4} g_{L\underline{L}}, \quad g^{\underline{L}\underline{L}} = \frac{1}{4} g_{LL}, \quad g^{LA} = -\frac{1}{2} g_{\underline{L}A}, \quad g^{\underline{L}A} = -\frac{1}{2} g_{LA}, \quad g^{AB} = g_{AB}, \quad (2.4)$$

here and below A, B denote any of the vectors S_1, S_2 .

Through the whole paper, we denote by

$$\bar{\partial} = \{L, S_1, S_2\}, \quad \text{which are tangent to the outgoing light cone } \{x_0^2 = \sum_{\alpha=1}^3 x_\alpha^2\},$$

and

$$\partial_q = \frac{1}{2}(\partial_r - \partial_t), \quad \partial_p = \frac{1}{2}(\partial_r + \partial_t).$$

Based on the preparations above, next we cite or establish some inequalities which will be used frequently.

Lemma 2.1. (See [20] Lemma 3.1) *For any smooth function u*

$$(1 + t + r)|\bar{\partial}u| \leq C|Zu|, \quad (2.5)$$

$$(1 + |t - r|)^k |\partial^k u| \leq C \sum_{0 \leq I \leq k} |Z^I u|, \quad (2.6)$$

$$(1 + t + r)|\partial u| \leq Cr|\partial_q u| + C|Zu|, \quad (2.7)$$

$$|\bar{\partial}^2 u| + r^{-1}|\bar{\partial}u| \leq \frac{C}{r} \sum_{0 \leq I \leq 2} \frac{|Z^I u|}{1 + t + r}, \quad \text{where } |\bar{\partial}^2 u|^2 = \sum_{S, T \in \mathbb{T}} |STu|^2, \quad \mathbb{T} = \{L, S_1, S_2\}. \quad (2.8)$$

Next, as in Lemma 3.2 of [20], we look for some “good derivatives” so that $\tilde{\square}_g u$ in (1.3) can be approximated well. To this end, we set

$$D^{ij} = g^{ij} - c^{ij} - E^{ij}, \quad (2.9)$$

where

$$E^{ij} = \sum_{k=0}^3 e_k^{ij} \partial_k u. \quad (2.10)$$

Let $L_1^i = -\frac{1}{2}g_{L\bar{L}}L^i - \frac{1}{4}g_{LL}\bar{L}^i + g_{LA}A^i$ (this definition is the same as that in (2.9) of [20, Lemma 2.1]), where $g_{LA}A^i = g_{LS_1}S_1^i + g_{LS_2}S_2^i$. Together with (2.3) and (2.9), this yields

$$L_1^i = L^i - \frac{1}{2}D_{L\bar{L}}L^i - \frac{1}{4}D_{LL}\bar{L}^i + D_{LA}A^i - \frac{1}{2}E_{L\bar{L}}L^i - \frac{1}{4}E_{LL}\bar{L}^i + E_{LA}A^i \quad (2.11)$$

In addition, we introduce another somewhat different L_2^i from that in (3.8) of [20] due to the appearance of E^{ij} in the coefficients $g^{ij}(u, \partial u)$:

$$L_2^i = L^i - \frac{1}{4}D_{LL}\bar{L}^i - \frac{1}{4}E_{LL}\bar{L}^i. \quad (2.12)$$

Similar to Lemma 3.2 of [20], we have

Lemma 2.2. Assume $|D| \leq \frac{1}{4}$ and $|E| \leq \frac{1}{8}$, then

$$|(2L_1^i\partial_i + \frac{\ell}{r})(r\partial_q u) - r\tilde{\square}_g u| \leq \frac{C}{1+t+r} \sum_{0 \leq I \leq 2} |Z^I u|, \quad (2.13)$$

where $\ell = \bar{t}rD + D_{L\bar{L}} - \frac{1}{2}D_{LL} + \bar{t}rE + E_{L\bar{L}} - \frac{1}{2}E_{LL}$ with $\bar{t}rD = \delta^{AB}D_{AB}$, $\bar{t}rE = \delta^{AB}E_{AB}$ and δ^{AB} the Kronecher delta function. If assume also that

$$|D_{LL}| + |D_{LA}| + |D_{AA}| + |D_{L\bar{L}}| \leq \frac{1+|t-r|}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \quad (2.14)$$

and

$$|E_{LL}| + |E_{LA}| + |E_{AA}| + |E_{L\bar{L}}| \leq \frac{1}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a, \quad a \geq 0, \quad (2.15)$$

then

$$|2L_2^i\partial_i(r\partial_q u) - r\tilde{\square}_g u| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{0 \leq I \leq 2} |Z^I u|. \quad (2.16)$$

Proof. At first, we point out that although the conclusions in Lemma 2.2 are rather analogous to the ones in Lemma 3.2 of [20], we still give the detailed proof since the coefficients $g^{ij}(u, \partial u)$ and the operator ℓ in (2.13) are somewhat different from the corresponding ones in [20].

Note that

$$\bar{t}rg = \delta^{AB}g_{AB} = 2 + \bar{t}rD + \bar{t}rE \quad \text{and} \quad 2L_1^i\partial_i r = 2 - D_{L\bar{L}} + \frac{1}{2}D_{LL} - E_{L\bar{L}} + \frac{1}{2}E_{LL},$$

which derives $\ell = \bar{t}rg - 2L_1^i\partial_i r$. Then it follows from a direct computation that

$$\begin{aligned} & (2L_1^i\partial_i + \frac{\ell}{r})(r\partial_q u) - r\tilde{\square}_g u \\ &= 2L_1^i\partial_i(r\partial_q u) - 2(L_1^i\partial_i r)\partial_q u + (\bar{t}rg)\partial_q u - r\tilde{\square}_g u \\ &= r(2L_1^i\partial_i\partial_q u + \frac{\bar{t}rg}{r}\partial_q u - \tilde{\square}_g u). \end{aligned} \quad (2.17)$$

By $|2L_1^i \partial_i \partial_q u + \frac{\bar{t}r}{r} \partial_q u - \tilde{\square}_g u| \leq C \sum_{1 \leq k \leq 2} r^{k-2} |\bar{\partial}^k u|$ (see (2.16) of [20, Lemma 2.2]) together with (2.8), thus we have

$$|(2L_1^i \partial_i + \frac{\ell}{r})(r \partial_q u) - r \tilde{\square}_g u| \leq C \sum_{0 \leq I \leq 2} \frac{|Z^I u|}{1+t+r}, \quad (2.18)$$

which means that (2.13) holds.

In addition, it follows from $L^i \partial_i r = 1$ and $A^i \partial_i r = 0$ that

$$\begin{aligned} & 2L_2^i \partial_i (r \partial_q u) - r \tilde{\square}_g u \\ &= 2L_1^i \partial_i (r \partial_q u) + 2(\frac{1}{2} D_{L\bar{L}} L^i - D_{LA} A^i + \frac{1}{2} E_{L\bar{L}} L^i - E_{LA} A^i) \partial_i (r \partial_q u) - r \tilde{\square}_g u \\ &\equiv I + II + III, \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} I &= (2L_1^i \partial_i + \frac{\ell}{r})(r \partial_q u) - r \tilde{\square}_g u, \\ II &= -(\bar{t}r D - \frac{1}{2} D_{L\bar{L}} + \bar{t}r E - \frac{1}{2} E_{L\bar{L}}) \partial_q u \\ III &= r(D_{L\bar{L}} + E_{L\bar{L}}) L^i \partial_i \partial_q u - 2r(D_{LA} + E_{LA}) A^i \partial_i \partial_q u. \end{aligned}$$

Since the first term I has been estimated in (2.18), we only need to treat II and III in (2.19). By (2.14)-(2.15) and (2.6), one easily knows

$$|II| \leq C \frac{1+|t-r|}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a |\partial u| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a |Zu|. \quad (2.20)$$

On the other hand, it also follows from $L^i \partial_i \omega^j = 0$ and (2.5)-(2.6) that

$$\begin{aligned} r|L^i \partial_i (\underline{L}^j \partial_j u)| &= r|L^i \underline{L}^j \partial_i \partial_j u| \leq Cr |\bar{\partial} \partial u| \leq C |Z \partial u| \leq C \sum_{0 \leq I \leq 1} |\partial Z^I u| \\ &\leq \frac{C}{1+|t-r|} \sum_{0 \leq I \leq 2} |Z^I u|, \end{aligned}$$

which derives

$$r|L^i \partial_i \partial_q u| = \frac{r}{2} |L^i \partial_i (\underline{L}^j \partial_j u)| \leq \frac{C}{1+|t-r|} \sum_{0 \leq I \leq 2} |Z^I u|. \quad (2.21)$$

Similarly, one has

$$r|A^i \partial_i \partial_q u| \leq \frac{C}{1+|t-r|} \sum_{0 \leq I \leq 2} |Z^I u|. \quad (2.22)$$

Thus, collecting (2.21)-(2.22) together with (2.14)-(2.15) yields

$$|III| \leq \frac{C}{1+|t-r|} (|D_{L\bar{L}}| + |E_{L\bar{L}}| + |D_{LA}| + |E_{LA}|) \sum_{0 \leq I \leq 2} |Z^I u|$$

$$\leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{0 \leq I \leq 2} |Z^I u|. \quad (2.23)$$

Finally, substituting (2.18), (2.20) and (2.23) into (2.19) yields (2.16). \square

As in [20] and [4], in order to obtain the sharp decay of the solution u to (1.3), we will adopt the idea of integration along the integral curves for the eikonal equation of (1.3) (see (2.34) below) so that the usual phase $r - t$ can be replaced and further the decay estimates of u on a curved background can be treated conveniently.

Let $X_\lambda(s) = (X_\lambda^0(s), X_\lambda^1(s), X_\lambda^2(s), X_\lambda^3(s)) \in \mathbb{R}^{1+3}$ ($\lambda = 1, 2$) be the backward integral curve of the vector fields $L_\lambda = L_\lambda^i \partial_i$. Namely, the components of $X_\lambda(s)$ satisfy

$$\begin{cases} \frac{d}{ds} X_\lambda^i(s) = L_\lambda^i(X_\lambda(s)), & s \leq 0, \\ X_\lambda(0) = (t, x), \end{cases} \quad (2.24)$$

where $(t, x) \in H \equiv \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 : \frac{t}{2} < |x| < \frac{3t}{2}\}$ which is near the light cone. Let $s_\lambda < 0$ be the largest number such that $X_\lambda(s_\lambda) \in \partial H$ and $X_\lambda(s) \in H$ for $s > s_\lambda$, where ∂H represents the boundary of H . Define $\tau_\lambda = \tau_\lambda(t, x) = X_\lambda^0(s_\lambda)$. When we assume $|D| \leq 1/4$ and $|E| \leq 1/8$, then the corresponding integral curves will intersect ∂H obviously. Next we establish some results similar to the ones in Lemma 4.1 of [20] so that the decay estimates of the first order derivatives ∂u can be controlled.

Lemma 2.3. Assume $|D| \leq 1/16$, $|E| \leq 1/32$ and

$$\int_0^T (\|D(t, \cdot)\|_{L^\infty(H_t)} + \|E(t, \cdot)\|_{L^\infty(H_t)}) \frac{dt}{1+t} \leq 1, \quad (2.25)$$

where $H_t \equiv \{x \in \mathbb{R}^3 : t/2 < |x| < 3t/2\}$, then

$$\begin{aligned} (1+t+r)|\partial u(t, x)| &\leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty} \\ &+ C \int_{\tau_1}^t \left((1+\tau) \|\tilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} + \sum_{0 \leq I \leq 2} (1+\tau)^{-1} \|Z^I u(\tau, \cdot)\|_{L^\infty(H_\tau)} \right) d\tau. \end{aligned} \quad (2.26)$$

If we assume $|D| \leq 1/16$, $|E| \leq 1/32$, and for some constant $a \geq 0$

$$\begin{aligned} |D_{LL}| + |D_{LA}| + |D_{AA}| + |D_{L\bar{L}}| &\leq \frac{1}{4} \frac{1+|t-r|}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \quad \text{in } H, \\ |E_{LL}| + |E_{LA}| + |E_{AA}| + |E_{L\bar{L}}| &\leq \frac{1}{4} \frac{1}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a, \end{aligned} \quad (2.27)$$

then

$$(1+t+r)|\partial u(t, x)| \leq C \sup_{\tau_2 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty}$$

$$+ C \int_{\tau_2}^t \left((1 + \tau) \|\tilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} + \sum_{0 \leq I \leq 2} (1 + \tau)^{-1+a} \|(1 + |q(\tau, \cdot)|)^{-a} Z^I u(\tau, \cdot)\|_{L^\infty(H_\tau)} \right) d\tau, \quad (2.28)$$

where $q(t, x) = r - t$.

Remark 2.1. By comparison with the assumptions in Lemma 4.1 of [20], we have posed an extra decay assumption on the term $|E_{LL}| + |E_{LA}| + |E_{AA}| + |E_{L\bar{L}}|$ in (2.27). Notice that there is no such an assumption (2.27) in [20] since there coefficients $g^{ij}(u)$ depend only on the solution itself.

Proof. By (2.7) it suffices to prove that $\phi = r\partial_q u$ can be controlled by the right hand side of (2.26) or (2.28). To this end, we will divide the proof process into the following two cases of $(t, x) \notin H$ and $(t, x) \in H$ separately.

Case A. $(t, x) \notin H$

In this case, one has $|t - r| \geq t/2$. This, together with $r \leq 1 + t$ and $\underline{L} = \frac{S - \omega^\alpha \Gamma_{0\alpha}}{r - t}$, yields

$$|\phi(t, x)| = \frac{1}{2} |r \underline{L} u| \leq C(1 + 2|t - r|) |\underline{L} u| \leq C \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty}. \quad (2.29)$$

Case B. $(t, x) \in H$

By the characteristics method and (2.24), we have

$$\frac{d}{ds} (\phi(X_1(s)) e^{G_1(s)}) = \frac{1}{2} e^{G_1(s)} \left((2L_1^i \partial_i + \frac{\ell}{r}) \phi \right) (X_1(s)), \quad (2.30)$$

where $G_1(s) = -\frac{1}{2} \int_s^0 \frac{\ell(X_1(\sigma))}{r(X_1(\sigma))} d\sigma$, and the quantity ℓ has been defined in (2.13) of Lemma 2.2.

If $\tau_1 = \tau_1(t, x) \geq 1$, then $\tau \geq \tau_1 \geq 1$ for $s \geq s_1$ and further $r \geq \frac{\tau}{2} \geq \frac{1 + \tau}{4}$ for $(\tau, x) \in X_1(s) \in H$. Thus, one has under the assumption (2.25)

$$\begin{aligned} |G_1(s)| &\leq \frac{1}{2} \int_s^0 \left| \frac{\ell}{r}(X_1(\sigma)) \right| d\sigma \leq \frac{1}{2} \int_{s_1}^0 \left| \frac{\ell}{r}(X_1(\sigma)) \right| d\sigma \\ &\leq C \int_{\tau_1}^t \frac{\|D(\tau, \cdot)\|_{L^\infty(H_\tau)} + \|E(\tau, \cdot)\|_{L^\infty(H_\tau)}}{1 + \tau} d\tau \leq C. \end{aligned} \quad (2.31)$$

In addition, by $t = X_1^0(s)$ and $dX_1^0(s)/ds = L_1^0$, where $L_1^0 = 1 - 1/4 D_{LL} - 1/2 D_{L\bar{L}} - 1/4 E_{LL} - 1/2 E_{L\bar{L}}$, we can derive $\frac{1}{2} \leq dt/ds \leq 2$ under the assumptions $|D| \leq 1/16$ and $|E| \leq 1/32$. Hence integrating (2.30) from s_1 to 0 together with (2.31) and Lemma 2.2 yields

$$|\phi(t, x)| \leq |\phi(X_1(s_1))| e^{G_1(s_1)} + \frac{1}{2} \int_{s_1}^0 \left| e^{G_1(s)} \left| \left((2L_1^i \partial_i + \frac{\ell}{r}) \phi \right) (X_1(s)) \right| \right| ds$$

$$\begin{aligned}
&\leq C|\phi(X_1(s_1))| + C \int_{s_1}^0 \left| \left((2L_1^i \partial_i + \frac{\ell}{r}) \phi \right) (X_1(s)) \right| ds \\
&\leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty} + C \int_{\tau_1}^t ((1+\tau) \|\tilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} \\
&\quad + \sum_{0 \leq I \leq 2} (1+\tau)^{-1} \|Z^I u(\tau, \cdot)\|_{L^\infty(H_\tau)}) d\tau,
\end{aligned} \tag{2.32}$$

here we have used the following fact

$$|\phi(X_1(s_1))| \leq C|(r-t)\underline{L}u(X_1(s_1))| \leq C \sum_{0 \leq I \leq 1} |Z^I u(X_1(s_1))| \leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty}$$

due to $X_1(s_1) \in \partial H$ (which means $r = t/2$ or $3t/2$, and then $r = t - r$ or $3(r - t)$ holds), $\partial_q = -\frac{1}{2}\underline{L}$ and $\underline{L} = \frac{S - \omega^\alpha \Gamma_{0\alpha}}{r - t}$.

If $\tau_1 < 1$, then we know that there exists \bar{s} with $s_1 < \bar{s} \leq 0$ such that $X_1^0(\bar{s}) = 1$ since t is decreasing along the backward integral curve $X_1(s)$ and s is an increasing function of t . Thus, as in (2.32), integrating (2.30) from \bar{s} to 0 yields

$$\begin{aligned}
|\phi(t, x)| &\leq |\phi(X_1(\bar{s}))| e^{G_1(\bar{s})} + \frac{1}{2} \int_{\bar{s}}^0 |e^{G_1(s)}| \left| \left((2L_1^i \partial_i + \frac{\ell}{r}) \phi \right) (X_1(s)) \right| ds \\
&\leq C|\phi(X_1(\bar{s}))| + \int_{\bar{s}}^0 \left| \left((2L_1^i \partial_i + \frac{\ell}{r}) \phi \right) (X_1(s)) \right| ds \\
&\leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty} + C \int_1^t ((1+\tau) \|\tilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} \\
&\quad + \sum_{0 \leq I \leq 2} (1+\tau)^{-1} \|Z^I u(\tau, \cdot)\|_{L^\infty(H_\tau)}) d\tau \\
&\leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty} + C \int_{\tau_1}^t ((1+\tau) \|\tilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} \\
&\quad + \sum_{0 \leq I \leq 2} (1+\tau)^{-1} \|Z^I u(\tau, \cdot)\|_{L^\infty(H_\tau)}) d\tau,
\end{aligned} \tag{2.33}$$

here we have applied the facts of $r \leq 1 + X_1^0(\bar{s}) = 2$ and

$$|\phi(X_1(\bar{s}))| \leq 2|\partial_q u| \leq C \sum_{0 \leq I \leq 1} |Z^I u(X_1(\bar{s}))| \leq C \sup_{\tau_1 \leq \tau \leq t} \sum_{0 \leq I \leq 1} \|Z^I u(\tau, \cdot)\|_{L^\infty}.$$

Therefore, combining (2.29) with (2.32)-(2.33) yields (2.26).

To prove (2.28), one can follow similarly from integrating $\frac{d}{ds} \phi(X_2(s)) = (L_2^i \partial_i \phi)(X_2(s))$ and applying (2.16) in Lemma 2.2. \square

As in [20], let $\rho = \rho(t, x)$ be constant along the integral curves of the vector field $L_2 = L_2^i \partial_i$ close to the light cone and equal to $r - t$ outside a neighborhood of the forward light cone so that

the usual phase $r - t$ can be replaced by $\rho(t, x)$ for the equation (1.3) on a curved background. Namely, $\rho = \rho(t, x)$ satisfies

$$L_2^i \partial_i \rho = 0 \quad \text{when} \quad |t - r| \leq t/2, \quad \rho = r - t \quad \text{when} \quad |t - r| \geq t/2. \quad (2.34)$$

We notice that (τ_2, \bar{x}) is the first intersection point of the backward integral curve with $|r - t| = t/2$, then by the definition of ρ , we have

$$|\rho(t, x)| = |\rho(\tau_2, \bar{x})| = \tau_2/2 \leq t/2 \quad \text{when} \quad |t - r| \leq t/2.$$

In addition, we will take $\rho = \rho(q, p, \omega)$ as a function of $q = r - t, p = r + t$ and $\omega = x/|x|$. As can be shown below, $0 < \partial_q \rho = \rho_q < \infty$ holds, then q can be also considered an invertible function of ρ for fixed (p, ω) and $\partial_q = \rho_q \partial_\rho$. We also note that

$$[L_2^i \partial_i, \partial_q] = -\frac{\partial_q D_{LL}}{2} \partial_q - \frac{\partial_q E_{LL}}{2} \partial_q \quad \text{and} \quad [L_2^i \partial_i, \partial_\rho] = 0. \quad (2.35)$$

Here we point out that by comparison with (5.5) in [20], there is an extra troublesome term $\frac{\partial_q E_{LL}}{2} \partial_q$ in $[L_2^i \partial_i, \partial_q]$ of (2.35), which should be specially paid attention since $\partial_q E_{LL}$ contains the second order derivatives $\partial^2 u$ but $\partial_q D_{LL}$ only contains the first order derivatives ∂u . With respect to the technical treatments on the coefficient $\partial_q E_{LL}$, one can refer to §3 (see (3.9)-(3.11)) below. Next we list an equivalence relation between ρ and q , and the estimate on $\partial_\rho \partial_q \rho$ which are completely analogous to the ones in Proposition 5.1 and Lemma 5.2 of [20] respectively.

Lemma 2.4. *Let $\rho(t, x)$ be defined as in (2.34), and assume that D_{LL} and E_{LL} satisfy*

$$|\partial_q D_{LL}| + |\partial_q E_{LL}| \leq \frac{C\varepsilon}{1+t} \frac{1}{(1+|\rho|)^\nu} \quad (2.36)$$

and

$$|D_{LL}| + |E_{LL}| \leq C\varepsilon \frac{1+|q|}{1+t} \quad (2.37)$$

for some $\nu \geq 0$. Then

$$\left(\frac{1+t}{1+|\rho|} \right)^{-C\varepsilon V(\rho)} \leq \rho_q \leq \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)} \quad \text{with } V(\rho) = (1+|\rho|)^{-\nu} \quad (2.38)$$

and

$$\left(\frac{1+t}{1+|\rho|} \right)^{-C\varepsilon} \leq \frac{1+|q|}{1+|\rho|} \leq \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon}. \quad (2.39)$$

In addition, if one further assumes

$$|\partial_q^2 D_{LL}| + |\partial_q^2 E_{LL}| \leq \frac{C\rho_q \varepsilon}{1+t} \frac{1}{(1+|\rho|)^{1+\nu}} \quad (2.40)$$

for some $\nu \geq 0$, then

$$|\partial_\rho \partial_q \rho| \leq \frac{C\rho_q \varepsilon}{(1+|\rho|)^{1+\nu}} \ln \left| \frac{1+t}{1+|\rho|} \right|. \quad (2.41)$$

Remark 2.2. By $E^{ij} = \sum_{k=1}^3 e_k^{ij} \partial_k u$, it seems that the decay property on the time t of $\partial_q E_{LL}$ in (2.36) should coincide with that of the second order derivative $\partial^2 u$, namely, $|\partial_q E_{LL}| \leq C|\partial^2 u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}$ (see (3.61) of Proposition 3.9 below). However, thanks to the null condition property of $e_k^{ij} \partial_k u \partial_{ij} u$, we can show that $\partial_q E_{LL}$ will admit better decay rate of $(1+t)^{-1}$ (see (3.9) in §3). This is one of the key points that we can show the global existence of the solution to (1.3).

Remark 2.3. Here we point out that we only need the assumption (2.36) to derive (2.38). And the assumption (2.37) suffices to get (2.39).

Proof. The proof procedures are completely similar to those in Proposition 5.1 and Lemma 5.2 of [20] under the assumptions (2.36)-(2.37) and (2.40), then we omit the details here. \square

At the end of this section, we will give two useful inequalities when the related null condition holds.

Lemma 2.5. Assume $\sum_{k=0}^3 e_k^{ij} \omega_i \omega_j \omega_k \equiv 0$ (i.e., the null condition) holds, then for smooth functions u, v supported in $|x| \leq 1+t$, we have

$$\left| \sum_{k=0}^3 e_k^{ij} \partial_k u \partial_{ij}^2 v \right| \leq C(|\bar{Z}u| |\partial^2 v| + |\partial u| |\bar{Z} \partial v|) \quad (2.42)$$

and

$$\left| \sum_{k=0}^3 e_k^{ij} \partial_k u \partial_{ij}^2 v \right| \leq C(1+t)^{-1}(|Zu| |\partial^2 v| + |\partial u| |Z \partial v|), \quad (2.43)$$

where $\bar{Z} = \{\partial_1 + \omega_1 \partial_t, \partial_2 + \omega_2 \partial_t, \partial_3 + \omega_3 \partial_t\}$.

Proof. By the null condition $\sum_{k=0}^3 e_k^{ij} \omega_i \omega_j \omega_k \equiv 0$, then

$$\begin{aligned} & \sum_{k=0}^3 e_k^{ij} \partial_k u \partial_{ij}^2 v \\ &= \sum_{k=0}^3 \left(e_k^{ij} (\partial_k u + \omega_k \partial_t u) \partial_{ij}^2 v - e_k^{ij} \omega_k \partial_t u (\partial_i + \omega_i \partial_t) \partial_j v + e_k^{ij} \omega_k \omega_i \partial_t u (\partial_j + \omega_j \partial_t) \partial_t v \right. \\ & \quad \left. - e_k^{ij} \omega_k \omega_i \omega_j \partial_t u \partial_t^2 v \right) \\ &= \sum_{k=0}^3 \left(e_k^{ij} (\partial_k u + \omega_k \partial_t u) \partial_{ij}^2 v - e_k^{ij} \omega_k \partial_t u (\partial_i + \omega_i \partial_t) \partial_j v + e_k^{ij} \omega_k \omega_i \partial_t u (\partial_j + \omega_j \partial_t) \partial_t v \right), \end{aligned}$$

which derives (2.42).

In addition, by $\partial_k u + \omega_k \partial_t u = t^{-1} \Gamma_{0k} u - \omega_k t^{-1} (r - t) \partial_t u$ ($k = 1, 2, 3$) and (2.6), we then have

$$\left| \sum_{k=0}^3 e_k^{ij} \partial_k u \partial_{ij}^2 v \right| \leq C(1+t)^{-1} (|Zu| |\partial^2 v| + |\partial u| |Z\partial v|).$$

Therefore, the proof of Lemma 2.5 is completed. \square

§3. The sharp decay estimate of the solution u to (1.3)

As in [20], we assume that the solution u of (1.3) admits the following weak decay estimate

$$|Z^I u| \leq M\varepsilon(1+t)^{-\nu}, \quad |I| \leq N-3, \quad M\varepsilon \leq 1 \quad (3.1)$$

for some large $N \geq 8$, $\frac{1}{2} < \nu < 1$ and a positive constant M . From this, we manage to derive the strong decay estimate of u and further obtain the more precise energy estimates in §5 and §6. Below we denote $C > 0$ by a generic constant depending only on M .

By the finite propagation speed property for the wave equation (1.3), then

$$u(t, x) \equiv 0 \quad \text{for } r \geq 1+t. \quad (3.2)$$

In addition, by scaling we may assume from now on for $0 < c_0 \ll 1$

$$|D| + |E| \leq c_0(|u| + |\partial u|). \quad (3.3)$$

First, we establish the strong decay estimates of u and ∂u than the ones in (3.1).

Lemma 3.1. *Assume that the solution u to (1.3) satisfies (3.1)-(3.3). Then*

$$|u| \leq \begin{cases} C\varepsilon(1+t)^{-1}(1+|q|), \\ C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{1-\nu-C\varepsilon} \end{cases} \quad (3.4)$$

and

$$|\partial u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-\nu}, \quad (3.5)$$

where $q = r - t$, and the function ρ has been defined in (2.34).

Proof. At first, we prove the first estimate in (3.4). By (3.1) and (3.3), then for any $T > 0$ and small ε

$$\begin{aligned} & \int_0^T (\|D(t, \cdot)\|_{L^\infty(H_t)} + \|E(t, \cdot)\|_{L^\infty(H_t)}) \frac{dt}{1+t} \leq c_0 \int_0^\infty \frac{(|u| + |\partial u|)}{1+t} dt \\ & \leq c_0 M\varepsilon \int_0^\infty (1+t)^{-1-\nu} dt \leq 1, \end{aligned}$$

where $H_t = \{x \in \mathbb{R}^3 : \frac{t}{2} < |x| < \frac{3t}{2}\}$. Together with (2.26), (3.1) and $\tilde{\square}_g u = 0$, this yields

$$(1+t+r)|\partial u| \leq C\varepsilon$$

and

$$|\partial u| \leq C\varepsilon(1+t+r)^{-1} \leq C\varepsilon(1+t)^{-1}. \quad (3.6)$$

Thus, the first estimate of (3.4) follows from integrating (3.6) from $r = 1+t$ where $u = 0$,

$$|u(t, x)| = |u(t, r, \omega)| = \left| - \int_r^{1+t} \partial_{r'} u(t, r', \omega) dr' \right| \leq \int_r^{1+t} C\varepsilon(1+t)^{-1} dr' \leq C\varepsilon(1+t)^{-1}(1+|q|). \quad (3.7)$$

Next, we show (3.5). In fact, by (3.6) and (3.7), we have

$$|\partial D| + |E| + (1+|q|)^{-1}|D| \leq \frac{C\varepsilon}{1+t}$$

or

$$|\partial D| + |E| + (1+|q|)^{-1}|D| \leq \frac{C\varepsilon}{1+t+r}.$$

Therefore, by choosing $a = \mu = 0$ in Lemma 4.2 of [20] (note that we have established Lemma 2.3 in §2, then the conclusion analogous to Lemma 4.2 of [20] follows by an easy computation), we can obtain

$$\begin{aligned} (1+t+r)(1+|\rho(t, x)|^\nu)|\partial u(t, x)| &\leq C \sup_{0 \leq \tau \leq t} (1+\tau)^\nu \sum_{0 \leq I \leq 2} \|Z^I u(\tau, \cdot)\|_{L^\infty} \\ &\quad + C \int_0^t (1+\tau) \|(1+|\rho(\tau, \cdot)|)^\nu \widetilde{\square}_g u(\tau, \cdot)\|_{L^\infty(H_\tau)} d\tau \\ &= C \sup_{0 \leq \tau \leq t} (1+\tau)^\nu \sum_{0 \leq I \leq 2} \|Z^I u(\tau, \cdot)\|_{L^\infty}. \end{aligned}$$

This, together with (3.1), yields (3.5).

Finally, we show the second estimate in (3.4). It is noted that we have by (3.5)

$$|\partial_q D_{LL}| + |E_{LL}| \leq \frac{C\varepsilon}{1+t} \frac{1}{(1+|\rho|)^\nu}. \quad (3.8)$$

To derive the second estimate in (3.4), we will apply Lemma 2.4 and (3.5) to derive such a kind of estimate $|\partial u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-\nu}$ so that the estimate of u can be obtained by integrating from $r = t+1$ where $u = 0$. To this end, we require to verify the assumptions (2.36)-(2.37) of Lemma 2.4. In fact, by (3.7)-(3.8), we only need to verify

$$|\partial_q E_{LL}| \leq \frac{C\varepsilon}{1+t} \frac{1}{(1+|\rho|)^\nu}. \quad (3.9)$$

Indeed, by the null condition $\sum_{k=0}^3 e_k^{ij} \omega_i \omega_j \omega_k \equiv 0$ and $L_i = \omega_i$ (due to $L_0 = -L^0 = -1$ and $L_\alpha = \omega_\alpha$), we have

$$\partial_q E_{LL} = -\frac{1}{2} \underline{L}^m \partial_m (E^{ij} L_i L_j)$$

$$\begin{aligned}
&= -\frac{1}{2}\underline{L}^m \partial_m \left(\sum_{k=0}^3 e_k^{ij} \partial_k u L_i L_j \right) \\
&= -\frac{1}{2}\underline{L}^m \sum_{k=0}^3 e_k^{ij} (\partial_m \partial_k u) L_i L_j \\
&= -\frac{1}{2}\underline{L}^m e_0^{ij} (\partial_t \partial_m u + \omega_m \partial_t^2 u) L_i L_j - \frac{1}{2}\underline{L}^m \sum_{\alpha=1}^3 e_\alpha^{ij} (\partial_\alpha \partial_m u - \omega_\alpha \omega_m \partial_t^2 u) L_i L_j \\
&\quad - \frac{1}{2}\underline{L}^m \omega_m (-e_0^{ij} L_i L_j + e_\alpha^{ij} \omega_\alpha L_i L_j) \partial_t^2 u \\
&= -\frac{1}{2}\underline{L}^m e_0^{ij} (\partial_t \partial_m u + \omega_m \partial_t^2 u) L_i L_j - \frac{1}{2}\underline{L}^m \sum_{\alpha=1}^3 e_\alpha^{ij} (\partial_\alpha \partial_m u - \omega_\alpha \omega_m \partial_t^2 u) L_i L_j, \quad (3.10)
\end{aligned}$$

here we have used the crucial null condition to derive $-e_0^{ij} L_i L_j + e_\alpha^{ij} \omega_\alpha L_i L_j \equiv 0$.

In addition, one also has for $\alpha = 1, 2, 3$

$$\partial_t \partial_\alpha u + \omega_\alpha \partial_t^2 u = t^{-1} \Gamma_{0\alpha} \partial_t u - t^{-1} \omega_\alpha q \partial_t^2 u,$$

$$\partial_1 \partial_\alpha u - \omega_1 \omega_\alpha \partial_t^2 u = t^{-1} (\Gamma_{01} \partial_\alpha u - \omega_1 \Gamma_{0\alpha} \partial_t u) - t^{-1} \omega_1 q (\partial_t \partial_\alpha u - \omega_\alpha \partial_t^2 u),$$

and $\partial_2 \partial_\alpha, \partial_3 \partial_\alpha$ proceed similarly. According to this and (3.10), together with (3.1) and the facts of $|\partial^2 u| \leq C(1+|q|)^{-1}|Z\partial u|$ and $|\rho| \leq Ct$, we arrive at

$$|\partial_q E_{LL}| \leq C(1+t)^{-1}(|Z\partial u| + |q||\partial^2 u|) \leq C(1+t)^{-1}|Z\partial u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-\nu}, \quad (3.11)$$

that is, (3.9) is shown.

By (3.8) and (3.9), we hence conclude that the assumptions (2.36)-(2.37) of Lemma 2.4 hold. Then by (2.39) in Lemma 2.4, one has

$$|\partial u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-\nu} \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-\nu}.$$

From this and $\int_r^{1+t} (1+|r'-t|)^{-\nu} dr' \leq C(1+|q|)^{1-\nu}$, we have

$$\begin{aligned}
|u(t, x)| &= \left| - \int_r^{1+t} \partial_{r'} u(t, r', \omega) dr' \right| \leq C\varepsilon(1+t)^{-1+C\varepsilon} \int_r^{1+t} (1+|r'-t|)^{-\nu} dr' \\
&\leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{1-\nu}. \quad (3.12)
\end{aligned}$$

Using (2.39) again for (3.12), we then get the second estimate in (3.4). \square

Next we derive the decay estimate of $\partial^2 u$. Before doing this, as in §6 of [20], we require to establish some Lemmas so that $L_2^i \partial_i (r \partial_q^2 u)$ and $L_2^i \partial_i (r \partial_\rho \partial_q u)$ can be suitably approximated by $r \partial_q \widetilde{\square}_g u$ and $r \partial_\rho \widetilde{\square}_g u$ respectively (in fact, $r \partial_q \widetilde{\square}_g u = 0$ and $r \partial_\rho \widetilde{\square}_g u = 0$ hold by the equation (1.3)). If so, integrating along the integral curve of $L_2^i \partial_i$, one can obtain the decay of $\partial^2 u$. In this process, we should specially pay attention to the terms E and ∂E which include ∂u and $\partial^2 u$ respectively.

Lemma 3.2. Assume that for some constant $a \geq 0$

$$|D| \leq \frac{1 + |t - r|}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a \quad (3.13)$$

and

$$|\partial D| + |E| \leq \frac{1}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a. \quad (3.14)$$

Then

$$\begin{aligned} |2L_2^i \partial_i (r \partial_q^2 u) + r(\partial_q D_{LL} + \partial_q E_{LL}) \partial_q^2 u - r \partial_q \tilde{\square}_g u| &\leq \frac{C}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a \sum_{0 \leq I \leq 2} |\partial Z^I u| \\ &\quad + C |\partial E| |\partial Z u|. \end{aligned} \quad (3.15)$$

Remark 3.1. Compared with (3.15) of Lemma 3.3 in [20], here an extra term $|\partial E| |\partial Z u|$ appears in the right hand side of (3.15). This term cannot be controlled by $\frac{C}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a \times \sum_{0 \leq I \leq 2} |\partial Z^I u|$ directly since we have no estimate on ∂E so far. In addition, the coefficient of $\partial_q^2 u$ in the left hand side of (3.15) is also different from that in (3.15) of [20] due to the appearance of $\partial_q E_{LL}$.

Proof. It follows from (2.4) and a direct computation (or see [20, Lemma 2.2]) that for $f^{ij} = \partial_m (D^{ij} + E^{ij})$

$$|f^{ij} \partial_i \partial_j u - f_{LL} \partial_q^2 u| \leq C |\bar{\partial} \partial u|, \quad (3.16)$$

where $C = \sup_{i,j} |f^{ij}|$ and $\bar{\partial} = \{L, S_1, S_2\}$.

In addition, one has

$$\begin{aligned} f_{LL} &= f^{ij} L_i L_j = \partial_m (D^{ij} + E^{ij}) L_i L_j \\ &= \partial_m D_{LL} - D^{ij} (\partial_m L_i) L_j - D^{ij} L_i (\partial_m L_j) + \partial_m E_{LL} - E^{ij} (\partial_m L_i) L_j - E^{ij} L_i (\partial_m L_j). \end{aligned} \quad (3.17)$$

Note that $|\partial_m L_i| \leq C/r$ and $|\partial_q^2 u| \leq C(1 + |t - r|)^{-1} |\partial Z u|$ hold. This, together with (2.5), (3.16)-(3.17) and the assumptions (3.13)-(3.14), yields

$$\begin{aligned} &|\partial_m (D^{ij} + E^{ij}) \partial_i \partial_j u - (\partial_m D_{LL} + \partial_m E_{LL}) \partial_q^2 u| \\ &\leq C(|\partial D| + |\partial E|) |\bar{\partial} \partial u| + |(D^{ij} (\partial_m L_i) L_j + D^{ij} L_i (\partial_m L_j) + E^{ij} (\partial_m L_i) L_j + E^{ij} L_i (\partial_m L_j)) \partial_q^2 u| \\ &\leq \frac{C}{r} \frac{1}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a \sum_{0 \leq I \leq 1} |\partial Z^I u| + \frac{C}{1 + t + r} |\partial E| |\partial Z u|. \end{aligned} \quad (3.18)$$

On the other hand, if we use $\partial_m u$ in (2.16) instead of u , then

$$|2L_2^i \partial_i (r \partial_q \partial_m u) - r \tilde{\square}_g \partial_m u| \leq \frac{C}{1 + t + r} \left| \frac{1 + t + r}{1 + |t - r|} \right|^a \sum_{0 \leq I \leq 2} |Z^I \partial_m u|. \quad (3.19)$$

Since

$$\partial_m \tilde{\square}_g u = \tilde{\square}_g \partial_m u + (\partial_m D^{ij}) \partial_i \partial_j u + (\partial_m E^{ij}) \partial_i \partial_j u, \quad (3.20)$$

by (3.19)-(3.20) and (3.18) we have

$$\begin{aligned} & |2L_2^i \partial_i (r \partial_q \partial_m u) + r (\partial_m D_{LL} + \partial_m E_{LL}) \partial_q^2 u - r \partial_m \tilde{\square}_g u| \\ & \leq |2L_2^i \partial_i (r \partial_q \partial_m u) - r \tilde{\square}_g \partial_m u| + r | - \partial_m (D^{ij} + E^{ij}) \partial_i \partial_j u + (\partial_m D_{LL} + \partial_m E_{LL}) \partial_q^2 u| \\ & \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{0 \leq I \leq 2} |Z^I \partial_m u| + C |\partial E| |\partial Z u|. \end{aligned} \quad (3.21)$$

Using $\partial_q = -\frac{1}{2}\underline{L} = -\frac{1}{2}\underline{L}^m \partial_m$ in (3.21), then we complete the proof of (3.15). \square

Lemma 3.3. Assume

$$|\partial D| + |E| + (1+|q|)^{-1}|D| \leq \frac{C\varepsilon}{1+t+r}, \quad (3.22)$$

then

$$|2L_2^i \partial_i (r \partial_\rho \partial_q u) - r \partial_\rho \tilde{\square}_g u| \leq \frac{C\rho_q^{-1}}{1+t+r} \sum_{0 \leq I \leq 2} |\partial Z^I u| + C\rho_q^{-1} |\partial E| |\partial Z u| \quad (3.23)$$

and

$$|2L_2^i \partial_i (r \partial_\rho \phi) - r \rho_q^{-1} \tilde{\square}_g \phi| \leq c_1 \varepsilon |\partial_\rho \phi| + \frac{C\rho_q^{-1}}{1+t+r} \sum_{0 \leq I \leq 2} |\partial Z^I \phi|. \quad (3.24)$$

Remark 3.2. Similar to Remark 3.1, by comparison with (6.11) of Lemma 6.2 in [20], here an extra term $\rho_q^{-1} |\partial E| |\partial Z \phi|$ also appears in the right hand side of (3.23).

Proof. Due to $L_2^i \partial_i \rho = 0$, $2L_2^i \partial_i \partial_q \rho = -\partial_q D_{LL} \partial_q \rho - \partial_q E_{LL} \partial_q \rho$ and $2L_2^i \partial_i \rho_q^{-1} = \rho_q^{-1} \partial_q D_{LL} + \rho_q^{-1} \partial_q E_{LL}$, it follows that

$$2L_2^i \partial_i (r \rho_q^{-1} \partial_q^2 u) = \rho_q^{-1} [2L_2^i \partial_i (r \partial_q^2 u) + (\partial_q D_{LL} + \partial_q E_{LL}) r \partial_q^2 u] \quad (3.25)$$

and further

$$2L_2^i \partial_i (r \partial_q^2 u) = 2\rho_q L_2^i \partial_i (r \rho_q^{-1} \partial_q^2 u) - r (\partial_q D_{LL} + \partial_q E_{LL}) \partial_q^2 u. \quad (3.26)$$

Applying Lemma 3.2 with $a = 0$ for (3.26) yields

$$|2\rho_q L_2^i \partial_i (r \rho_q^{-1} \partial_q^2 u) - r \partial_q \tilde{\square}_g u| \leq \frac{C}{1+t+r} \sum_{0 \leq I \leq 2} |\partial Z^I u| + C |\partial E| |\partial Z u|,$$

which derives (3.23).

(3.24) follows from (2.16) with $a = 0$ and $|\partial_q E_{LL}| \leq C\varepsilon(1+t)^{-1}$ directly. \square

Based on Lemma 3.2 and Lemma 3.3, we now establish the strong decay estimate of second order derivatives $\partial^2 u$.

Lemma 3.4. Assume that the solution u to (1.3) satisfies (3.1)-(3.3). Then for small $\varepsilon > 0$

$$|\partial^2 u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}\rho_q. \quad (3.27)$$

Proof. When $(t, x) \notin H$, $|q| \geq t/2$ and $\rho = q$ and further $\frac{1+t}{1+|\rho|} = O(1)$. Then it follows from this, (2.6) and (3.1) that

$$\begin{aligned} |\partial^2 u| &\leq C(1+|q|)^{-2} \sum_{0 \leq I \leq 2} |Z^I u| \\ &\leq C\varepsilon(1+|\rho|)^{-2}(1+t)^{-\nu} \\ &\leq C\varepsilon(1+|\rho|)^{-1-\nu} \left(\frac{1+t}{1+|\rho|} \right)^{1-\nu} (1+t)^{-1} \\ &\leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}, \end{aligned}$$

which implies (3.27) holds.

We now consider the case of $(t, x) \in H$. Notice that ∂^2 can be expressed as the combinations of ∂_q^2 , $\bar{\partial}^2$ and $\bar{\partial}\partial_q$ with bounded coefficients, and thus

$$|\partial^2 u| \leq C(|\partial_q^2 u| + |\bar{\partial}^2 u| + |\bar{\partial}\partial_q u|). \quad (3.28)$$

We now analyze each term in the right hand side of (3.28). To obtain the estimate of $\partial_q^2 u$, we first study $L_2^i \partial_i (r\rho_q^{-1} \partial_q^2 u)$. For this aim, applying (3.23), Lemma 2.4 and (3.1) yields

$$\begin{aligned} |2L_2^i \partial_i (r\rho_q^{-1} \partial_q^2 u)| &\leq \frac{C\rho_q^{-1}}{1+t} \sum_{0 \leq I \leq 2} |\partial Z^I u| + C\rho_q^{-1} |\partial E| |\partial Z u| \\ &\leq \frac{C\rho_q^{-1}}{(1+t)(1+|q|)} \sum_{0 \leq I \leq 3} |Z^I u| + \frac{C\rho_q^{-1}}{(1+|q|)^2} |ZE| |Z^2 u| \\ &\leq \frac{C\varepsilon}{(1+t)^{1+\nu-C\varepsilon}(1+|\rho|)^{1+C\varepsilon}} + \frac{C\varepsilon}{(1+t)^{2\nu-C\varepsilon}(1+|\rho|)^{2+C\varepsilon}}. \end{aligned} \quad (3.29)$$

Since $\nu > \frac{1}{2} + C\varepsilon$ holds for small ε , integrating (3.29) from $r = t/2$ (at this place, $t \sim |\rho|$ and $|r\rho_q^{-1} \partial_q^2 u| \leq C\varepsilon(1+|\rho|)^{-\nu-1}$) yields

$$|r\rho_q^{-1} \partial_q^2 u| \leq C\varepsilon(1+|\rho|)^{-\nu-1}$$

and further

$$|\partial_q^2 u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-\nu-1}\rho_q. \quad (3.30)$$

Next we estimate $\bar{\partial}^2 u$. Without loss of generality, we assume $t > 1$. In this case, $r > \frac{t}{2} > \frac{1+t}{4}$ holds for $(t, x) \in H$. Therefore, by (2.8) and (3.1), we have

$$|\bar{\partial}^2 u| \leq \frac{C}{r} \sum_{0 \leq I \leq 2} \frac{|Z^I u|}{1+t+r}$$

$$\begin{aligned}
&\leq C\varepsilon(1+t)^{-1}(1+t)^{-1-\nu} \\
&\leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}\rho_q.
\end{aligned} \tag{3.31}$$

Similarly, we have

$$|\bar{\partial}\partial_q| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}\rho_q.$$

This, together with (3.30)-(3.31) and (3.28), yields (3.27). \square

Next we establish the strong decay estimate on Zu . To this end, we require to calculate the commutators of vector fields Z with $\tilde{\square}_g = \square + D^{ij}\partial_i\partial_j + \sum_{k=0}^3 e_k^{ij}\partial_k u\partial_i\partial_j$. Below $G(u, v)$ denotes the various bilinear form analogous to $e_k^{ij}\partial_k u\partial_{ij}^2 v$, which satisfies the null condition. It follows from Lemma 6.6.5 in [11] and a direct computation that

$$\begin{aligned}
Z\tilde{\square}_g\phi &= Z\square\phi + Z(D^{ij}\partial_i\partial_j\phi) + ZG(u, \phi) \\
&= \square Z\phi - C_Z\square\phi + D^{ij}\partial_i\partial_j Z\phi + (ZD^{ij})\partial_i\partial_j\phi + 2D^{ij}C_{Zi}^l\partial_l\partial_j\phi + G(Zu, \phi) \\
&\quad + G(u, Z\phi) + G(u, \phi) \\
&= \tilde{\square}_g Z\phi - C_Z\tilde{\square}_g\phi + C_Z D^{ij}\partial_i\partial_j\phi + 2D^{ij}C_{Zi}^l\partial_l\partial_j\phi + (ZD^{ij})\partial_i\partial_j\phi + G(Zu, \phi) + G(u, \phi),
\end{aligned}$$

where C_Z and C_{Zi}^l are some suitable constants. Set $\hat{Z} = Z + C_Z$, then we have

$$\tilde{\square}_g Z\phi = \hat{Z}\tilde{\square}_g\phi - (ZD^{ij} + 2C_{Zi}^l D^{lj} + C_Z D^{ij})\partial_i\partial_j\phi + G(Zu, \phi) + G(u, \phi) \tag{3.32}$$

and further

$$\tilde{\square}_g Z^I\phi = \hat{Z}^I\tilde{\square}_g\phi + \sum_{J+K\leq I, K<I} C_{JKij}^{I\ lm}(Z^J D^{ij})\partial_l\partial_m Z^K\phi + \sum_{J+K\leq I, K<I} G(Z^J u, Z^K\phi), \tag{3.33}$$

where $C_{JKij}^{I\ lm}$ are constants.

More generally, we can have

$$\begin{aligned}
\tilde{\square}_g \partial^k Z^I\phi &= \partial^k \hat{Z}^I\tilde{\square}_g\phi + \sum_{J+K\leq I, s+n=k, n+K<k+I} C_{JKij}^{I\ lm}(\partial^s Z^J D^{ij})\partial_l\partial_m \partial^n Z^K\phi \\
&\quad + \sum_{J+K\leq I, s+n=k, n+K<k+I} G(\partial^s Z^J u, \partial^n Z^K\phi).
\end{aligned} \tag{3.34}$$

Thus, by (3.34) and the equation (1.3), we have for u

$$\begin{aligned}
|\tilde{\square}_g \partial^k Z^I u| &\leq C \sum_{J+K\leq I, K<I} |Z^J u| |\partial^2 \partial^k Z^K u| + \sum_{s+n=k, J+K\leq I} |\partial \partial^s Z^J u| |\partial \partial^n Z^K u| \\
&\quad + \sum_{J+K\leq I, s+n=k, n+K<k+I} |G(\partial^s Z^J u, \partial^n Z^K\phi)|.
\end{aligned} \tag{3.35}$$

Before estimating Zu , we require to cite the result in Lemma 6.3 of [20] for reader's convenience. Here we point out that although the form of L_2^i in our paper is somewhat different from

that in [20], by minor modification on the proof of Lemma 6.3 in [20] (still integrating along the integral curve of L_2 and applying Gronwall's inequality), we have

Lemma 3.5. *Assume for some constant $\nu > \frac{1}{2}$*

$$(1 + |\rho|)|\partial_\rho \phi| + |\phi| \leq C\varepsilon(1 + |\rho|)^{-\nu} \quad \text{when } |t - r| = t/2 \quad \text{or} \quad t + r \leq 2$$

and

$$\phi = 0 \quad \text{when } r > 1 + t \quad \text{and} \quad t > 0.$$

Assume also that for $(t, x) \in H$

$$|L_2^i \partial_i(r \partial_\rho \phi)| \leq C\varepsilon \left(\frac{|\phi|}{1 + |\rho|} + |\partial_\rho \phi| \right) + \frac{C\varepsilon^2}{(1 + t)^{1-C\varepsilon}(1 + |\rho|)^{\nu+C\varepsilon}} + \frac{C\varepsilon(1 + |\rho|)^{-C\varepsilon}}{(1 + t)^{1+\nu-C\varepsilon}}.$$

Then

$$|\phi|(1 + |\rho|)^{-1} + |\partial_\rho \phi| \leq C\varepsilon(1 + t)^{-1+C\varepsilon}(1 + |\rho|)^{-\nu}.$$

With respect to the decay estimate of Zu , we have

Lemma 3.6. *Assume that the solution u to (1.3) satisfies (3.1)-(3.3). Then*

$$|Zu| \leq \begin{cases} C\varepsilon(1 + t)^{-1+C\varepsilon}(1 + |\rho|)^{1-\nu}, \\ \varepsilon(1 + t)^{-1}(|q| + (1 + t)^{C\varepsilon}). \end{cases} \quad (3.36)$$

Proof. Due to $\tilde{\square}_g u = 0$, we have by (3.32) and Lemma 2.5

$$|\tilde{\square}_g Z\phi| \leq C(|Zu| + |u|)|\partial^2 u| + C(1 + t)^{-1} \left(\sum_{1 \leq I \leq 2} |Z^I u| |\partial^2 u| + \sum_{0 \leq I \leq 1} |\partial Z^I u| |Z\partial u| \right). \quad (3.37)$$

On the other hand, if we use Zu instead of ϕ in (3.24) and combine (3.37), then

$$\begin{aligned} |L_2^i \partial_i(r \partial_\rho Zu)| &\leq Cr(|Zu| + |u|) \frac{|\partial^2 u|}{\rho_q} + C\rho_q^{-1} \left(\sum_{1 \leq I \leq 2} |Z^I u| |\partial^2 u| + \sum_{0 \leq I \leq 1} |\partial Z^I u| |Z\partial u| \right) \\ &\quad + C\varepsilon |\partial_\rho Zu| + \frac{C\rho_q^{-1}}{1 + t} \sum_{0 \leq J \leq 3} |\partial Z^J u|. \end{aligned}$$

Therefore it follows from (3.1), (3.4)-(3.5), (3.27) and (2.38)-(2.39) that

$$|L_2^i \partial_i(r \partial_\rho Zu)| \leq C\varepsilon \left(\frac{|Zu|}{1 + |\rho|} + |\partial_\rho Zu| \right) + \frac{C\varepsilon^2}{(1 + t)^{1-C\varepsilon}(1 + |\rho|)^{2\nu+C\varepsilon}} + \frac{C\varepsilon(1 + |\rho|)^{-C\varepsilon}}{(1 + t)^{1+\nu-C\varepsilon}}. \quad (3.38)$$

Note that when $|t - r| = t/2$

$$|Zu| + (1 + |\rho|)|\partial_\rho Zu| = |Zu| + (1 + |q|)|\partial Zu| \leq C \sum_{0 \leq I \leq 2} |Z^I u| \leq C\varepsilon(1 + t)^{-\nu} \leq C\varepsilon(1 + |\rho|)^{-\nu}. \quad (3.39)$$

Therefore, by (3.38)-(3.39), one knows that all the assumptions of Lemma 3.5 are fulfilled, then it follows that

$$|Zu|(1+|\rho|)^{-1} + |\partial_\rho Zu| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{-\nu}.$$

Together with $|\rho| \leq Ct$ and $(1+t)^{C\varepsilon}(1+|q|)^{1-\nu} \leq (1+t)^{C\varepsilon/\nu} + (1+|q|)$, this yields (3.36). \square

Next, we establish the decay estimate of $\partial^k u$ for $1 \leq k \leq N-4$.

Lemma 3.7. *Assume that the solution u to (1.3) satisfies (3.1)-(3.3). For $1 \leq k \leq N-4$, we have*

$$|\partial^k u| \leq \frac{C\varepsilon}{1+t}(1+|\rho|)^{1-k-\nu} \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)}, \quad (3.40)$$

where $V(\rho) = (1+|\rho|)^{-\nu}$.

Proof. We will show (3.40) by induction method. If $k=1$, we have already proved (3.40) by (3.5) of Lemma 3.1. Assume that (3.40) holds for $k \leq n \leq N-5$, we will prove (3.40) for $k=n+1$. Taking ∂^n on two hand sides of $\tilde{\square}_g u = c^{ij}\partial_i\partial_j u + D^{ij}\partial_i\partial_j u + E^{ij}\partial_i\partial_j u$ yields

$$\tilde{\square}_g \partial^n u = - \sum_{m+k=n, m \geq 1} (\partial^m D^{ij}) \partial_i \partial_j \partial^k u - \sum_{m+k=n, m \geq 1} (\partial^m E^{ij}) \partial_i \partial_j \partial^k u. \quad (3.41)$$

Substituting (3.5) and (3.40) with $k \leq n$ into (3.41) yields

$$\begin{aligned} & |\tilde{\square}_g \partial^n u| \\ & \leq C|\partial u| \sum_{m=n+1} |\partial^m u| + C \sum_{2 \leq m \leq n, 0 \leq k \leq n-2, m+k=n} |\partial^m u| |\partial^2 \partial^k u| + \sum_{m+k=n, m \geq 1} |G(\partial^m u, \partial^k u)| \\ & \leq \frac{C\varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{m=n+1} |\partial^m u| + \frac{C\varepsilon^2}{(1+t)^2(1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)} \\ & \quad + \sum_{m+k=n, m \geq 1} |G(\partial^m u, \partial^k u)|. \end{aligned} \quad (3.42)$$

Note that

$$\begin{aligned} \sum_{m+k=n, m \geq 1} |G(\partial^m u, \partial^k u)| &= \sum_{k=n-1} |G(\partial u, \partial^k u)| + |G(\partial^n u, u)| \\ &+ \sum_{m+k=n, 2 \leq m \leq n-1, 1 \leq k \leq n-2} |G(\partial^m u, \partial^k u)|. \end{aligned} \quad (3.43)$$

By Lemma 2.5, (3.1), (2.6), (2.39) and the fact of $|\rho| \leq Ct$, we have

$$\begin{aligned} & \sum_{k=n-1} |G(\partial u, \partial^k u)| + |G(\partial^n u, u)| \\ & \leq C(1+t)^{-1}(|Z\partial u| |\partial^{n+1} u| + |\partial^2 u| |Z\partial^n u|) \\ & \leq \frac{C\varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{m=n+1} |\partial^m u| + \frac{C\varepsilon^2}{(1+t)^2(1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)}. \end{aligned} \quad (3.44)$$

In addition, it follows from (3.40) with $k \leq n$ that

$$\sum_{m+k=n, 2 \leq m \leq n-1, 1 \leq k \leq n-2} |G(\partial^m u, \partial^k u)| \leq \frac{C\varepsilon^2}{(1+t)^2(1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)}. \quad (3.45)$$

Substituting (3.43)-(3.45) into (3.42) yields

$$|\tilde{\square}_g \partial^n u| \leq \frac{C\varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{m=n+1} |\partial^m u| + \frac{C\varepsilon^2}{(1+t)^2(1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon V(\rho)}. \quad (3.46)$$

On the other hand, by (2.6), (2.39) and (3.1), one has

$$|Z^I \partial^n u| \leq C(1+|q|)^{-n} \sum_{J \leq n+I} |Z^J u| \leq C\varepsilon(1+|q|)^{-n}(1+t)^{-\nu} \leq C\varepsilon(1+t)^{-\nu+nC\varepsilon}(1+|\rho|)^{-n-nC\varepsilon}. \quad (3.47)$$

In terms of (3.46) and (3.47), we know that both the assumptions in Lemma 6.6 of [20] hold. Thus, it follows from Lemma 6.6 of [20] that (3.40) holds for $k = n + 1$. And the proof of Lemma 3.7 is completed. \square

Finally, we derive the decay estimate of $|\partial^k Z^I u|$ with $\max(1, k) + I \leq N - 4$.

Lemma 3.8. *Assume that the solution u to (1.3) satisfies (3.1)-(3.3). For $\max(1, k) + I \leq N - 4$, then we have*

$$|\partial^k Z^I u| \leq C(1+t)^{-1+C\varepsilon}(1+|q|)^{1-k-\nu}. \quad (3.48)$$

Proof. Thanks to (2.39), we just need to show

$$|\partial^k Z^I u| \leq C(1+t)^{-1+C\varepsilon}(1+|\rho|)^{1-k-\nu}. \quad (3.49)$$

Next we use the induction method to prove (3.49).

For $I = 0$ and all k , we have already proved (3.49) in (3.40) of Lemma 3.7.

Assume that (3.49) holds for $I \leq m - 1$ ($m \geq 1$) and all k , then we can prove (3.49) for $I = m$ and $k \leq 1$. In fact, by (3.20), we have for $I = m$

$$\begin{aligned} |\tilde{\square}_g Z^m u| &\leq C|Z^m u||\partial^2 u| + C \sum_{J+K \leq m, J \leq m-1, K \leq m-1} |Z^J u||\partial^2 Z^K u| \\ &\quad + \sum_{J+K \leq m, K \leq m-1} |G(Z^J u, Z^K u)|. \end{aligned} \quad (3.50)$$

Hence, by (3.24) applied to $\phi = Z^m u$ and (3.50), together with (3.5), (3.27) and (3.49) for $m - 1$, one has

$$|L_2^i \partial_i (r \partial_\rho Z^m u)| \leq \frac{Cr}{\rho_q} \left(|Z^m u||\partial^2 u| + \sum_{J+K \leq m, J \leq m-1, K \leq m-1} |Z^J u||\partial^2 Z^K u| \right)$$

$$\begin{aligned}
& + \sum_{J+K \leq m, K \leq m-1} |G(Z^J u, Z^K u)| \Big) + C\varepsilon |\partial_\rho Z^m u| + \frac{C\rho_q^{-1}}{1+t} \sum_{J \leq m+2} |Z^J u| \\
& \leq C\varepsilon \left(\frac{|Z^m u|}{1+|\rho|} + |\partial_\rho Z^m u| \right) + \frac{C\varepsilon(1+|\rho|)^{-C\varepsilon}}{(1+t)^{1+\nu-C\varepsilon}} \\
& \quad + \frac{C\varepsilon^2}{(1+t)^{1-C\varepsilon}(1+|\rho|)^{2\nu}} + \frac{Cr}{\rho_q} \sum_{J+K \leq m, K \leq m-1} |G(Z^J u, Z^K u)|. \tag{3.51}
\end{aligned}$$

Since

$$\sum_{J+K \leq m, K \leq m-1} |G(Z^J u, Z^K u)| = |G(Z^m u, u)| + \sum_{J \leq m-1, K \leq m-1} |G(Z^J u, Z^K u)|,$$

by Lemma 2.5, (3.1), (2.38), (3.27) or (3.49) for $m-1$, we have respectively

$$\begin{aligned}
Cr\rho_q^{-1}|G(Z^m u, u)| & \leq Cr\rho_q^{-1}(1+t)^{-1}(|Z^{m+1}u||\partial^2 u| + |\partial Z^m u||Z\partial u|) \\
& \leq \frac{C\varepsilon^2}{(1+t)^{1-C\varepsilon}(1+|\rho|)^{2\nu}} \tag{3.52}
\end{aligned}$$

and

$$Cr\rho_q^{-1} \sum_{J \leq m-1, K \leq m-1} |G(Z^J u, Z^K u)| \leq \frac{C\varepsilon^2}{(1+t)^{1-C\varepsilon}(1+|\rho|)^{2\nu}}. \tag{3.53}$$

Substituting (3.52)-(3.53) into (3.51) yields

$$|L_2^i \partial_i (r \partial_\rho Z^m u)| \leq C\varepsilon \left(\frac{|Z^m u|}{1+|\rho|} + |\partial_\rho Z^m u| \right) + \frac{C\varepsilon(1+|\rho|)^{-C\varepsilon}}{(1+t)^{1+\nu-C\varepsilon}} + \frac{C\varepsilon^2}{(1+t)^{1-C\varepsilon}(1+|\rho|)^{2\nu}}. \tag{3.54}$$

Then it follows from (3.54) and Lemma 3.5 that

$$|Z^m u|(1+|\rho|)^{-1} + |\partial_\rho Z^m u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{-\nu}, \tag{3.55}$$

which means that (3.49) holds for $I = m$ and $k \leq 1$.

Finally, assuming (3.49) for $I \leq m$ and all $k \leq n$, and (3.49) for $I \leq m-1$ and all k , we now show that (3.49) holds for $I = m \geq 1$ and $k = n+1 \geq 2$.

In this case, by (3.34)-(3.35) together with (3.49) for $k \leq n$ and $I \leq m$, and $k \leq n+2$ and $I \leq m-1$, one has

$$\begin{aligned}
|\tilde{\square}_g \partial^n Z^I u| & \leq C|\partial u| \sum_{m=n+1} |\partial^m Z^I u| + \sum_{J+K \leq I, k \leq n, m+k=n+2, m \leq n \text{ or } k \leq I-1} |\partial^m Z^J u| |\partial^k Z^K u| \\
& \quad + \sum_{m+k=n, J+K \leq I, k+K < n+I} |G(\partial^m Z^J u, \partial^k Z^K u)| \\
& \leq \frac{C\varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{m=n+1} |\partial^m Z^I u| + \frac{C\varepsilon^2}{(1+t)^{2-C\varepsilon}(1+|\rho|)^{n+2\nu}}, \tag{3.56}
\end{aligned}$$

here we point out that $\sum_{m+k=n, J+K \leq I, k+K < n+I} |G(\partial^m Z^J u, \partial^k Z^K u)|$ has been treated as in (3.43)-(3.45). From (3.56), as shown in Lemma 3.7, we can show that (3.49) holds for $I = m \geq 1$ and $k = n + 1 \geq 2$. By induction method and all the analysis above, we complete the proof of Lemma 3.8. \square

In summary, collecting Lemma 3.1, Lemma 3.4 and Lemma 3.6-Lemma 3.8, we arrive at **Proposition 3.9.** *Assume that u is the solution to (1.3) and (3.1)-(3.3) hold. Then*

$$|u| \leq \begin{cases} C\varepsilon(1+t)^{-1}(1+|q|), \\ C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{1-\nu-C\varepsilon}, \end{cases} \quad (3.57)$$

$$|\partial u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-\nu}, \quad (3.58)$$

$$|\partial^2 u| \leq C\varepsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}\rho_q, \quad (3.59)$$

$$|Zu| \leq \begin{cases} C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{1-\nu}, \\ C\varepsilon(1+t)^{-1}(|q| + (1+t)^{C\varepsilon}), \end{cases} \quad (3.60)$$

furthermore,

$$|\partial^k Z^I u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{1-k-\nu} \quad \text{for } \max(1, k) + I \leq N - 4. \quad (3.61)$$

§4. Weighted energy estimates for the nonlinear problem (1.3)

As in [4] and [20], we now establish the weighted energy estimates for the equation (1.3) so that the standard energy $E_N(t) = \sum_{I \leq N} \int |\partial Z^I u(t, x)|^2 dx \leq C\varepsilon^2(1+t)^{C\varepsilon}$ can be shown under the weak assumption $E_N(t) \leq C\varepsilon^2(1+t)^\delta$ for some small fixed constant $0 < \delta < \frac{1}{2}$. From this and the higher order energy estimates in §5, the validity of the weak decay estimate (3.1) of u can be proved in §6.

We will choose such a weight in the weighted energy of (1.3)

$$W = e^{\sigma(t)V(\rho)+\varphi(q)} \quad (4.1)$$

with

$$\begin{cases} \sigma(t) = \kappa\varepsilon \ln |1+t|, \\ V(\rho) = |\rho - 2|^{-\nu'}, \\ \varphi'(q) = (1+|q|)^{-3/2}, \end{cases}$$

where $\kappa > 0$ and $\nu' > 1/2$ are constants, the function $\rho(t, x)$ has been defined in (2.34), and $q = r - t$. In addition, $\rho \leq 1$ is known.

Now we state the weighted energy estimate in this section.

Proposition 4.1. *Assume that u is a solution to (1.3) and the notations g, c, E (corresponding to g^{ij}, c^{ij}, E^{ij} , respectively) have been defined in previous sections. In addition, the following assumptions are fulfilled*

$$|g - c - E| \leq \frac{1}{4}, \quad |E| < \frac{1}{8}, \quad |\partial(g - E)| \leq \frac{C\varepsilon}{1+t}, \quad (4.2)$$

$$(1 + |q|)^{-1} |Z\partial u| + |\partial^2 u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-1-\nu'}, \quad (4.3)$$

$$\rho_t < 0, \quad \frac{g^{ij}\rho_i\rho_j}{\rho_t(1+|\rho|)^{1+\nu'}} \geq -\frac{1}{\kappa\varepsilon(1+t)\ln(1+t)}, \quad (4.4)$$

where κ and $\nu' > 1/2$ are given in (4.1), then we have

$$\begin{aligned} & \int_{\Sigma_t} |\partial u|^2 W dx + \int_0^t \int_{\Sigma_\tau} (1+|q|)^{-3/2} |\bar{Z}u|^2 W dx d\tau \\ & \leq C \int_{\Sigma_0} |\partial u|^2 W dx + \int_0^t \int_{\Sigma_\tau} \frac{C\varepsilon}{1+\tau} |\partial u|^2 W dx d\tau + \int_0^t \int_{\Sigma_\tau} \frac{C(1+\tau)}{\varepsilon} |\tilde{\square}_g u|^2 W dx d\tau, \end{aligned} \quad (4.5)$$

where W is defined in (4.1), $\Sigma_t = \{x \in \mathbb{R}^3 : |x| \leq 1+t\}$, and $|\bar{Z}u|^2 = \sum_{\alpha=1}^3 (\partial_\alpha u + \omega_\alpha \partial_t u)^2$.

Remark 4.1. *The choice of the complicated weight function W in (4.1) is due to the following two reasons: First, the factor $e^{\sigma(t)V(\rho)}$ in W comes from the weight in [4] and [20]*

which is used to derive the weighted energy estimate for the wave equation $\sum_{i,j=0}^3 g^{ij}(u) \partial_{ij}^2 u = 0$.

Motivated by this, we intend to search a new weight W containing the factor $e^{\sigma(t)V(\rho)}$ so that

the weighted energy estimate for the wave equation $\sum_{i,j=0}^3 g^{ij}(u, \partial u) \partial_{ij}^2 u = 0$ can be derived.

Second, due to the appearance of the term $e_k^{ij} \partial_k u \partial_{ij}^2 u$ in (1.3) and the related null condition property, we want to add another factor $e^{\varphi(q)}$ in the new weight W . In this case, we may obtain the good controls for the troublesome terms $Q_2 \partial_t^2 u$ and I_1 in (4.17) and (4.19) below, which are

induced by the appearances of $\sum_{k=0}^3 e_k^{ij} \partial_k u$ in the coefficients $g^{ij}(u, \partial u)$. The so-called ‘‘good control’’ means that the corresponding integral can be finally absorbed by the left hand side of (4.5).

Remark 4.2. *By the way, if the null condition does not hold (for this case, we have actually proved the blowup of solution to (1.3) in [7-8]), that is, $\sum_{k=0}^3 \frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k +$*

$\frac{1}{2} e_k^{\alpha\beta} \omega_\alpha \omega_\beta \omega_k \not\equiv 0$ holds, then we can not obtain the energy estimate (4.5) since the troublesome term $|u_t^2 (\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k + \frac{1}{2} e_k^{\alpha\beta} \omega_\alpha \omega_\beta \omega_k) \partial_t^2 u|$ ($\leq C\varepsilon(1+\tau)^{-1+C\varepsilon} |\partial u|^2$ by the assumption (4.3))

will appear in the right hand side of $|I_1|$ in (4.19) below, which is not a good control term (only $C\varepsilon(1+\tau)^{-1}|\partial u|^2$ is a good control term by Gronwall's inequality).

Proof. Denote by $v_\alpha = \partial_\alpha v$ for $\alpha = 1, 2, 3$ and $v_t = \partial_t v = v_0$, then it follows from a direct computation that

$$\begin{aligned} W\partial_t u \widetilde{\square}_g u &= (g^{00}\partial_t^2 u \partial_t u + 2g^{0\alpha}\partial_0\partial_\alpha u \partial_t u + g^{\alpha\beta}\partial_\alpha\partial_\beta u \partial_t u)W \\ &= \frac{1}{2}\partial_t [(g^{00}u_t^2 - g^{\alpha\beta}u_\alpha u_\beta)W] - \frac{1}{2}(g^{00}u_t^2 - g^{\alpha\beta}u_\alpha u_\beta)\partial_t W \\ &\quad + \partial_\alpha [(g^{0\alpha}u_t^2 + g^{\alpha\beta}u_t u_\beta)W] - (g^{0\alpha}u_t^2 + g^{\alpha\beta}u_t u_\beta)\partial_\alpha W \\ &\quad + W \left(-\frac{1}{2}u_t^2\partial_t g^{00} - u_t^2\partial_\alpha g^{0\alpha} - u_\beta u_t\partial_\alpha g^{\alpha\beta} + \frac{1}{2}u_\alpha u_\beta\partial_t g^{\alpha\beta} \right). \end{aligned} \quad (4.6)$$

Integrating (4.6) over $x \in \mathbb{R}^3$ yields

$$\frac{1}{2}\frac{d}{dt} \int (-g^{00}u_t^2 + g^{\alpha\beta}u_\alpha u_\beta)W dx = - \int u_t W \widetilde{\square}_g u dx + \int I_1 W dx + \int I_2 dx, \quad (4.7)$$

where

$$\begin{aligned} I_1 &= -\frac{1}{2}u_t^2\partial_t g^{00} - u_t^2\partial_\alpha g^{0\alpha} - u_\beta u_t\partial_\alpha g^{\alpha\beta} + \frac{1}{2}u_\alpha u_\beta\partial_t g^{\alpha\beta} \\ &= \frac{1}{2}u_i u_j \partial_t g^{ij} - u_t u_j \partial_i g^{ij}, \\ I_2 &= -\frac{1}{2}(g^{00}u_t^2 - g^{\alpha\beta}u_\alpha u_\beta)\partial_t W - (g^{0\alpha}u_t^2 + g^{\alpha\beta}u_t u_\beta)\partial_\alpha W \\ &= \frac{1}{2}g^{ij}u_i u_j W_t - g^{ij}u_i u_t W_j. \end{aligned}$$

Next we analyze each term in the right hand side of (4.7). At first, we treat I_1 in (4.7). Since $g^{ij} = c^{ij} + D^{ij} + E^{ij}$ and $|\partial(g - E)| \leq \frac{C\varepsilon}{1+t}$, it follows that

$$\left| \frac{1}{2}u_i u_j \partial_t (c^{ij} + D^{ij}) - u_t u_j \partial_i (c^{ij} + D^{ij}) \right| \leq C\varepsilon(1+t)^{-1}|\partial u|^2. \quad (4.8)$$

In addition, by a direct computation, we have

$$\begin{aligned} &\frac{1}{2}u_i u_j \partial_t E^{ij} - u_t u_j \partial_i E^{ij} \\ &= \frac{1}{2}u_i u_j \sum_{k=0}^3 e_k^{ij} \partial_t \partial_k u - u_t u_j \sum_{k=0}^3 e_k^{ij} \partial_i \partial_k u \\ &= u_t^2 \sum_{k=0}^3 \left(-\frac{1}{2}e_k^{00} \partial_t \partial_k u - e_k^{\beta 0} \partial_\beta \partial_k u \right) + u_t u_\beta \left(-\sum_{k=0}^3 e_k^{\alpha\beta} \partial_\alpha \partial_k u \right) + \frac{1}{2}u_\alpha u_\beta \sum_{k=0}^3 e_k^{\alpha\beta} \partial_t \partial_k u. \end{aligned} \quad (4.9)$$

Substituting $\partial_t \partial_k u = \partial_t \partial_k u + \omega_k \partial_t^2 u - \omega_k \partial_t^2 u$ and $\partial_\beta \partial_k u = \partial_\beta \partial_k u - \omega_\beta \omega_k \partial_t^2 u + \omega_\beta \omega_k \partial_t^2 u$ into the first term in the right hand side of (4.9) yields

$$\begin{aligned}
& u_t^2 \sum_{k=0}^3 \left(-\frac{1}{2} e_k^{00} \partial_t \partial_k u - e_k^{\beta 0} \partial_\beta \partial_k u \right) \\
&= u_t^2 \sum_{k=0}^3 \left[-\frac{1}{2} e_k^{00} (\partial_t \partial_k u + \omega_k \partial_t^2 u) - e_k^{\beta 0} (\partial_\beta \partial_k u - \omega_\beta \omega_k \partial_t^2 u) \right] + u_t^2 \sum_{k=0}^3 \left(\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k \right) \partial_t^2 u \\
&= -e_0^{\beta 0} u_t^2 (\partial_\beta \partial_t u + \omega_\beta \partial_t^2 u) + u_t^2 \sum_{k=1}^3 \left[-\frac{1}{2} e_k^{00} (\partial_t \partial_k u + \omega_k \partial_t^2 u) - e_k^{\beta 0} (\partial_\beta \partial_k u - \omega_\beta \omega_k \partial_t^2 u) \right] \\
&\quad + u_t^2 \sum_{k=0}^3 \left(\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k \right) \partial_t^2 u. \tag{4.10}
\end{aligned}$$

Similarly, the last two terms in the right hand side of (4.9) admit the following expressions respectively

$$\begin{aligned}
u_t u_\beta \left(-\sum_{k=0}^3 e_k^{\alpha\beta} \partial_\alpha \partial_k u \right) &= -u_t u_\beta e_0^{\alpha\beta} (\partial_\alpha \partial_t u + \omega_\alpha \partial_t^2 u) - u_t u_\beta \sum_{k=1}^3 e_k^{\alpha\beta} (\partial_\alpha \partial_k u - \omega_\alpha \omega_k \partial_t^2 u) \\
&\quad - u_t u_\beta \sum_{k=0}^3 e_k^{\alpha\beta} \omega_\alpha \omega_k \partial_t^2 u, \tag{4.11}
\end{aligned}$$

$$\frac{1}{2} u_\alpha u_\beta \sum_{k=0}^3 e_k^{\alpha\beta} \partial_t \partial_k u = \frac{1}{2} u_\alpha u_\beta \sum_{k=1}^3 e_k^{\alpha\beta} (\partial_t \partial_k u + \omega_k \partial_t^2 u) - \frac{1}{2} u_\alpha u_\beta \sum_{k=0}^3 e_k^{\alpha\beta} \omega_k \partial_t^2 u. \tag{4.12}$$

Note that for $k = 1, 2, 3$ and by (4.3),

$$|\partial_t \partial_k u + \omega_k \partial_t^2 u| = |t^{-1} \Gamma_{0k} \partial_t u - \omega_k t^{-1} q \partial_t^2 u| \leq C(1+t)^{-1} |Z \partial u| \leq C\varepsilon(1+t)^{-1} \tag{4.13}$$

and

$$|\partial_\beta \partial_k u - \omega_\beta \omega_k \partial_t^2 u| = |t^{-1} (\Gamma_{0\beta} \partial_k u - \omega_\beta \Gamma_{0k} \partial_t u) - t^{-1} \omega_\beta q (\partial_t \partial_k u - \omega_k \partial_t^2 u)| \leq C\varepsilon(1+t)^{-1}. \tag{4.14}$$

Then substituting (4.10)-(4.14) into (4.9) derives

$$\frac{1}{2} u_i u_j \partial_t E^{ij} - u_t u_j \partial_i E^{ij} = Q_1 + Q_2 \partial_t^2 u, \tag{4.15}$$

where $|Q_1| \leq C\varepsilon(1+t)^{-1} |\partial u|^2$ and $Q_2 = \sum_{k=0}^3 u_t^2 \left(\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k \right) - u_t u_\beta e_k^{\alpha\beta} \omega_\alpha \omega_k - \frac{1}{2} u_\alpha u_\beta e_k^{\alpha\beta} \omega_k$. Here we point out that the term $Q_2 \partial_t^2 u$ should be specially treated as follows.

Set $X_\alpha = u_\alpha + \omega_\alpha u_t$, then $u_\alpha = X_\alpha - \omega_\alpha u_t$. Thanks to the null condition property, we can obtain

$$\begin{aligned}
Q_2 &= \sum_{k=0}^3 u_t^2 \left(\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k \right) - u_t (X_\beta - \omega_\beta u_t) e_k^{\alpha \beta} \omega_\alpha \omega_k - \frac{1}{2} (X_\alpha - \omega_\alpha u_t) (X_\beta - \omega_\beta u_t) e_k^{\alpha \beta} \omega_k \\
&= \sum_{k=0}^3 u_t^2 \left(\frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k + \frac{1}{2} e_k^{\alpha \beta} \omega_\alpha \omega_\beta \omega_k \right) - \frac{1}{2} X_\alpha X_\beta e_k^{\alpha \beta} \omega_k \\
&= -\frac{1}{2} \sum_{k=0}^3 X_\alpha X_\beta e_k^{\alpha \beta} \omega_k \quad \left(\text{due to } \sum_{k=0}^3 \frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k + \frac{1}{2} e_k^{\alpha \beta} \omega_\alpha \omega_\beta \omega_k \equiv 0 \right). \quad (4.16)
\end{aligned}$$

Thus, by (4.16) and (4.3), one has

$$|Q_2 \partial_t^2 u| = \frac{1}{2} \left| \sum_{k=0}^3 X_\alpha X_\beta e_k^{\alpha \beta} \omega_k \partial_t^2 u \right| \leq C \sum_{\alpha=1}^3 X_\alpha^2 |\partial_t^2 u| \leq C\varepsilon (1+t)^{-1+C\varepsilon} (1+|q|)^{-1-\nu'} |\bar{Z}u|^2. \quad (4.17)$$

Obviously, if the null condition does not hold here, then $\sum_{k=0}^3 \frac{1}{2} e_k^{00} \omega_k - e_k^{\beta 0} \omega_\beta \omega_k + \frac{1}{2} e_k^{\alpha \beta} \omega_\alpha \omega_\beta \omega_k \neq 0$ and we only obtain

$$|Q_2 \partial_t^2 u| \leq C\varepsilon (1+t)^{-1+C\varepsilon} (1+|q|)^{-1-\nu'} |\partial u|^2 + C\varepsilon (1+t)^{-1+C\varepsilon} (1+|q|)^{-1-\nu'} |\bar{Z}u|^2, \quad (4.18)$$

which means that the first term in the right hand side of (4.18) can not be absorbed globally by the term $\int_{\Sigma_t} |\partial u|^2 W dx$ of (4.5) (the concrete reason is: in this case, (4.5) becomes $\int_{\Sigma_t} |\partial u|^2 W dx + \dots \leq \int_0^t \int_{\Sigma_\tau} C\varepsilon (1+t)^{-1+C\varepsilon} |\partial u|^2 W dx d\tau + \dots$, then the crucial integral $\int_{\Sigma_t} |\partial u|^2 W dx$ cannot be uniformly controlled by Gronwall's inequality since $\int_0^\infty (1+t)^{-1+C\varepsilon} dt = \infty$).

Consequently, collecting (4.8), (4.15) and (4.17) yields

$$|I_1| \leq C\varepsilon (1+t)^{-1} |\partial u|^2 + C\varepsilon (1+t)^{-1+C\varepsilon} (1+|q|)^{-1-\nu'} |\bar{Z}u|^2. \quad (4.19)$$

Next let us deal with the term I_2 in (4.7). Set

$$W = \widetilde{W} e^{\varphi(q)}, \quad \text{where } \widetilde{W} = e^{\sigma(t)V(\rho)}.$$

Then it follows from a direct computation that

$$\begin{aligned}
I_2 &= \left(\frac{1}{2} g^{ij} u_i u_j - g^{i0} u_i u_t \right) \left((A\rho_t + B)W - \varphi'(q)W \right) - g^{i\alpha} u_i u_t \left(A\rho_\alpha W + \varphi'(q)\omega_\alpha W \right) \\
&= W(I_{21} + I_{22} + I_{23}) \quad (4.20)
\end{aligned}$$

with

$$I_{21} = \left(\left(\frac{1}{2} g^{ij} u_i u_j - g^{i0} u_i u_t \right) \rho_t - g^{i\alpha} u_i u_t \rho_\alpha \right) A,$$

$$I_{22} = \left(\frac{1}{2} g^{ij} u_i u_j - g^{i0} u_i u_t \right) B,$$

$$I_{23} = \varphi'(q) \left(-\frac{1}{2} g^{ij} u_i u_j + g^{i0} u_i u_t - g^{i\alpha} u_i u_t \omega_\alpha \right),$$

where $A = \frac{\kappa \nu' \varepsilon \ln |1+t|}{|\rho-2|^{1+\nu'}}$ and $B = \frac{\kappa \varepsilon}{(1+t)|\rho-2|^{\nu'}}$. We now treat each term in the right hand side of (4.20).

Since $|g-c-E| < 1/4$ and $|E| < 1/8$, we have $|g-c| < 1/2$, which means that the 3×3 matrix $(g^{\alpha\beta})_{\alpha,\beta=1}^3$ is nonnegative definite. Thus, as in (7.9) of [20], we have by (4.4)

$$I_{21} \leq -\frac{1}{2} A \frac{g^{ij} \rho_i \rho_j u_t^2}{\rho_t} \leq \frac{C\varepsilon}{1+t} |\partial u|^2. \quad (4.21)$$

On the other hand, a direct computation yields

$$\frac{1}{2} g^{ij} u_i u_j - g^{i0} u_i u_t = \frac{1}{2} (-g^{00} u_t^2 + g^{\alpha\beta} u_\alpha u_\beta). \quad (4.22)$$

In addition, it follows from $|D| < 1/4$ and $|E| < 1/8$ that

$$\frac{1}{2} |\partial u|^2 = \frac{1}{2} (u_t^2 + \delta^{\alpha\beta} u_\alpha u_\beta) \leq -g^{00} u_t^2 + g^{\alpha\beta} u_\alpha u_\beta \leq 2(u_t^2 + \delta^{\alpha\beta} u_\alpha u_\beta) = 2|\partial u|^2. \quad (4.23)$$

This, together with (4.22) and $\rho \leq 1$, yields

$$I_{22} \leq \frac{C\varepsilon}{1+t} |\partial u|^2. \quad (4.24)$$

We now treat the third term in (4.20). It is noted that

$$\begin{aligned} -\frac{1}{2} c^{ij} u_i u_j + c^{i0} u_i u_t - c^{i\alpha} u_i u_t \omega_\alpha &= -\frac{1}{2} (-u_t^2 + \sum_{\alpha=1}^3 u_\alpha^2) - u_t^2 - u_\alpha u_t \omega_\alpha \\ &= -\frac{1}{2} \sum_{\alpha=1}^3 (u_\alpha + \omega_\alpha u_t)^2 = -\frac{1}{2} |\bar{Z}u|^2. \end{aligned} \quad (4.25)$$

Additionally, due to $\varphi'(q) = (1+|q|)^{-3/2}$, it follows that

$$\begin{aligned} &\left| \varphi'(q) \left[-\frac{1}{2} (D^{ij} + E^{ij}) u_i u_j + (D^{i0} + E^{i0}) u_i u_t - (D^{i\alpha} + E^{i\alpha}) u_i u_t \omega_\alpha \right] \right| \\ &\leq C |\varphi'(q)| (|u| + |\partial u|) |\partial u|^2 \\ &\leq C \varepsilon (1+t)^{-1} (1+|q|) |\varphi'(q)| |\partial u|^2 \\ &\leq C \varepsilon (1+t)^{-1} |\partial u|^2. \end{aligned}$$

This, together with (4.25), yields

$$I_{23} \leq -\frac{1}{2} (1+|q|)^{-\frac{3}{2}} |\bar{Z}u|^2 + C \varepsilon (1+t)^{-1} |\partial u|^2. \quad (4.26)$$

Thus, by substituting (4.21), (4.24) and (4.26) into (4.20), we have

$$I_2 \leq \left(-\frac{1}{2}(1+|q|)^{-\frac{3}{2}}|\bar{Z}u|^2 + C\varepsilon(1+t)^{-1}|\partial u|^2 \right) W. \quad (4.27)$$

Noting that

$$\begin{aligned} \int_0^t \int (-g^{ij} \partial_i \partial_j u) u_t W dx d\tau &= \int_0^t \int -u_t \tilde{\square}_g u W dx d\tau \\ &\leq \int_0^t \frac{C\varepsilon}{1+\tau} \int_{\Sigma_\tau} |\partial u|^2 W dx d\tau + \frac{C}{\varepsilon} \int_0^t \int_{\Sigma_\tau} (1+\tau) |\tilde{\square}_g u|^2 W dx d\tau, \end{aligned} \quad (4.28)$$

then integrating (4.7) over $[0, t]$ and applying (4.19), (4.23) and (4.27)-(4.28), we can arrive at

$$\begin{aligned} &\frac{1}{4} \int_{\Sigma_t} |\partial u|^2 W dx + \frac{1}{2} \int_0^t \int_{\Sigma_\tau} \left[1 - C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-\nu'+1/2} \right] (1+|q|)^{-3/2} |\bar{Z}u|^2 W dx d\tau \\ &\leq \frac{1}{2} \int_{\Sigma_0} (-g^{00} u_t^2 + g^{\alpha\beta} u_\alpha u_\beta) W dx + \frac{C}{\varepsilon} \int_0^t \int_{\Sigma_\tau} (1+\tau) |\tilde{\square}_g u|^2 W dx d\tau + \int_0^t \int_{\Sigma_\tau} \frac{C\varepsilon}{1+\tau} |\partial u|^2 W dx \\ &\leq \int_{\Sigma_0} |\partial u|^2 W dx + \frac{C}{\varepsilon} \int_0^t \int_{\Sigma_\tau} (1+\tau) |\tilde{\square}_g u|^2 W dx d\tau + \int_0^t \int_{\Sigma_\tau} \frac{C\varepsilon}{1+\tau} |\partial u|^2 W dx. \end{aligned} \quad (4.29)$$

Due to $\nu' > \frac{1}{2}$ and $1 - C\varepsilon(1+t)^{-1+C\varepsilon} \geq 1/2$ for small $\varepsilon > 0$, we then obtain (4.5) from (4.29). \square

§5. Higher order energy estimates for the problem (1.3)

In this section, under the strong decay assumptions on the solution u to (1.3), which are given in Prop.3.9, we will give the higher order energy estimates for $E_{k,i}(t) = \sum_{0 \leq \mathbf{k} \leq k, 0 \leq I \leq i} \int |\partial \partial^{\mathbf{k}} Z^I u|^2 W dx$.

Before doing this, we show a weighted Poincaré lemma similar to that in Lemma 8.1 of [20].

Lemma 5.1. *Assume that W is defined in (4.1) with suitably large $\kappa > 0$ and*

$$|\partial_\rho \partial_r \rho| \leq \frac{C\varepsilon \ln |1+t|}{(1+|\rho|)^{1+\nu'}} \partial_r \rho, \quad 0 < \partial_r \rho < \infty. \quad (5.1)$$

Then for function u supported in $r \leq 1+t$,

$$\int \left(\frac{|u|}{1+|\rho|} \partial_r \rho \right)^2 W dx + \int \left(\frac{|u|}{1+|r-t|} \right)^2 W dx \leq C \int |\partial u|^2 W dx. \quad (5.2)$$

Proof. Although the proof of (5.1) is completely similar to that in Lemma 8.1 of [20], we will still give the details for reader's convenience, due to the different form of the weight W . Notice that we only require to treat the first integral in the left hand side of (5.2) since the second one is a special case of the first with $\rho = r - t$.

Taking integration by parts, we have

$$\begin{aligned} \int_0^\infty \left(\frac{|u|}{|\rho-2|} \partial_r \rho \right)^2 W r^2 dr &= \int_{-\infty}^1 \left(\frac{|u|}{|\rho-2|} \right)^2 \partial_r \rho W r^2 d\rho = \int_{-\infty}^1 |u|^2 \partial_r \rho W r^2 d \left(\frac{1}{|\rho-2|} \right) \\ &= -2 \int_{-\infty}^1 \frac{u}{|\rho-2|} \partial_\rho u \partial_r \rho W r^2 d\rho - \int_{-\infty}^1 \left(\frac{u}{|\rho-2|} \right)^2 |\rho-2| \partial_\rho (\partial_r \rho W r^2) d\rho. \end{aligned} \quad (5.3)$$

In addition, by (5.1), $1 + |\rho| \geq \frac{1}{2}|\rho-2|$ and suitably large κ , we have

$$\begin{aligned} \partial_\rho (\partial_r \rho W r^2) &= (\partial_\rho \partial_r \rho) W r^2 + \partial_r \rho [(\partial_\rho W) r^2 + 2r W \partial_\rho r] \\ &\geq -\frac{C\varepsilon \ln |1+t|}{(1+|\rho|)^{1+\nu'}} (\partial_r \rho) W r^2 + \partial_r \rho \left[\frac{\kappa \nu' \varepsilon \ln |1+t|}{|\rho-2|^{1+\nu'}} + (1+|q|)^{-3/2} (\rho_q)^{-1} \right] W r^2 + 2r W \\ &\geq (1+|q|)^{-3/2} (\rho_q)^{-1} r^2 W \partial_r \rho + 2r W \geq 0. \end{aligned}$$

This, together with (5.3) and Hölder inequality, yields

$$\int_0^\infty \left(\frac{|u|}{|\rho-2|} \partial_r \rho \right)^2 W r^2 dr \leq 2 \left(\int_0^\infty \left(\frac{|u|}{|\rho-2|} \partial_r \rho \right)^2 W r^2 dr \right)^{1/2} \left(\int_0^\infty (\partial_\rho u \partial_r \rho)^2 W r^2 dr \right)^{1/2}$$

and further

$$\int_0^\infty \left(\frac{|u|}{|\rho-2|} \partial_r \rho \right)^2 W r^2 dr \leq 4 \int_0^\infty (\partial_\rho u \partial_r \rho)^2 W r^2 dr = 4 \int_0^\infty (\partial_r u)^2 W r^2 dr \quad (5.4)$$

Integrating (5.4) over the angular variables and using $\frac{1}{2}|\rho-2| \leq 1+|\rho|$, we obtain (5.2). \square

For the energy

$$E_{k,i}(t) = \sum_{0 \leq I \leq i, 0 \leq \mathbf{k} \leq k} \int |\partial \partial^{\mathbf{k}} Z^I u|^2 W dx \quad (5.5)$$

where W is defined in (4.1) with suitably large κ , we have

Proposition 5.2. *Let $N \geq 8$ be fixed, $1/2 < \nu' < 1$ and $N' = [N/2] + 2$. Assume that u is a solution of (1.3) for $0 \leq t < T$ and*

$$|\partial u| \leq \frac{C\varepsilon}{1+t} \frac{1}{(1+|\rho|)^{\nu'}}, \quad (5.6)$$

$$|\partial^2 u| \leq \frac{C\varepsilon \rho_q}{1+t} \frac{1}{(1+|\rho|)^{1+\nu'}}, \quad (5.7)$$

$$|u| + |Zu| \leq C\varepsilon (1+t)^{-1+C\varepsilon} (1+|\rho|)^{1-\nu'}, \quad (5.8)$$

$$|\partial Z^I u| + (1+|q|)^{-1} |Z^I u| \leq \frac{C\varepsilon}{1+t} (1+t)^{C\varepsilon} (1+|q|)^{-\nu'} \quad \text{for } I \leq N', \quad (5.9)$$

Then for $k + i \leq N$, $k, i \geq 0$,

$$E_{k,i}(t) \leq CE_{k,i}(0) + \int_0^t \frac{C\varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + \int_0^t \frac{C\varepsilon}{(1+\tau)^{1-C\varepsilon}} (E_{k+1,i-1}(\tau) + E_{k-1,i}(\tau)) d\tau, \quad (5.10)$$

where $E_{-1,n} = 0$ and $E_{m,-1} = 0$.

Proof. We will use Proposition 4.1 and Lemma 5.1 to prove (5.10). To this end, we need to verify all the assumptions (4.2)-(4.4) of Proposition 4.1 and (5.1) of Lemma 5.1 respectively.

First, let us verify the assumptions (4.2) of Proposition 4.1. By applying (5.8) and the facts of $|\rho| \leq Ct$ and $\nu' > 1/2$, we have

$$|u| + |Zu| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|\rho|)^{1-\nu'} \leq C\varepsilon(1+t)^{-1/2+C\varepsilon}.$$

From this and (5.6), it is enough to assume $|D| \leq \frac{1}{4}$ and $|E| \leq \frac{1}{8}$ for sufficiently small $\varepsilon > 0$.

In addition, (5.6) derives $|\partial(g - E)| \leq \frac{C\varepsilon}{1+t}$ directly. Thus, (4.2) holds.

Second, it is obvious that (5.6) implies (2.37) holds. By Remark 2.3, we know that (2.39) is true. As in (3.11), one has $|\partial_q E_{LL}| \leq \frac{C\varepsilon}{1+t} \frac{1}{(1+|\rho|)^{\nu'}}$ by (5.9) and (2.39). This, together with (5.6), yields (2.38) of Lemma 2.4. Then it follows from this, (5.7) and (5.9) that

$$(1+|q|)^{-1}|Z\partial u| + |\partial^2 u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-1-\nu'},$$

which means that (4.3) holds.

Third, by $L_2^i \partial_i \rho = 0$, we have $\partial_p \rho = -\frac{1}{4}(D_{LL} + E_{LL})\partial_q \rho$ and

$$-\rho_t = -(\rho_p - \rho_q) = (1 + \frac{1}{4}D_{LL} + \frac{1}{4}E_{LL})\partial_q \rho > \frac{1}{2}\rho_q \geq \frac{1}{2}(\frac{1+t}{1+|\rho|})^{-C\varepsilon}.$$

From (5.6) and (5.8)-(5.9), we know that the assumption in Lemma 5.3 of [20] hold. Then as in (7.3) of [20], we can arrive at

$$\frac{g^{ij}\rho_i\rho_j}{\rho_t(1+|\rho|)^{1+\nu'}} \geq -\frac{1/(\kappa\nu')}{(1+t)\ln(1+t)},$$

which means that (4.4) holds.

Fourth, we verify (5.1) of Lemma 5.1. To this end, we intend to use Lemma 2.4 to derive (5.1). We now verify the assumption (2.40) of Lemma 2.4. Due to (5.7), we only need to verify that $\partial_q^2 E_{LL}$ satisfies (2.40). As in (3.10), by the null condition, we have

$$\partial_q E_{LL} = -\frac{1}{2}\underline{L}^m e_0^{ij}(\partial_t \partial_m u + \omega_m \partial_t^2 u) L_i L_j - \frac{1}{2}\underline{L}^m \sum_{\alpha=1}^3 e_\alpha^{ij}(\partial_\alpha \partial_m u - \omega_\alpha \omega_m \partial_t^2 u) L_i L_j. \quad (5.11)$$

For the first term in the right hand side of (5.11),

$$\left| \partial_q \left(-\frac{1}{2}\underline{L}^m e_0^{ij}(\partial_t \partial_m u + \omega_m \partial_t^2 u) L_i L_j \right) \right| = \left| -\frac{1}{2}\underline{L}^m e_0^{ij} L_i L_j \partial_q (\partial_t \partial_m u + \omega_m \partial_t^2 u) \right|$$

$$\begin{aligned}
&= \left| -\frac{1}{4} \underline{L}^\alpha e_0^{ij} L_i L_j (\partial_r - \partial_t) (t^{-1} \Gamma_{0\alpha} \partial_t u - \omega_\alpha t^{-1} q \partial_t^2 u) \right| \\
&\leq C t^{-1} (|Z \partial^2 u| + |\partial^2 u|) + C t^{-2} |Z \partial u| \\
&\leq C t^{-1} (1 + |q|)^{-1} \sum_{0 \leq I \leq 2} |\partial Z^I u| + C t^{-2} |Z \partial u|.
\end{aligned}$$

This, together with (5.9), (2.38)-(2.39) and the fact of $|q| \leq C t$, yields

$$\begin{aligned}
\left| \partial_q \left(-\frac{1}{2} \underline{L}^m e_0^{ij} (\partial_t \partial_m u + \omega_m \partial_t^2 u) L_i L_j \right) \right| &\leq C \varepsilon (1+t)^{-1} (1+|q|)^{-1-\nu'} (1+t)^{-1+C\varepsilon} \\
&\leq C \varepsilon (1+t)^{-1} (1+|\rho|)^{-1-\nu'} \rho_q.
\end{aligned} \tag{5.12}$$

The second term in the right hand side of (5.11) proceed similarly. Hence, we have from (5.11)-(5.12)

$$|\partial_q^2 E_{LL}| \leq C \varepsilon (1+t)^{-1} (1+|\rho|)^{-1-\nu'} \rho_q,$$

which means that the assumption (2.40) in Lemma 2.4 holds. Then by (2.41) of Lemma 2.4, one has

$$|\partial_\rho \partial_q \rho| \leq \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} \partial_q \rho \ln \left| \frac{1+t}{1+|\rho|} \right|. \tag{5.13}$$

Note that $\partial_r \rho = \partial_\rho \rho + \partial_q \rho = (1 - D_{LL}/4 - E_{LL}/4) \partial_q \rho$ due to $L_2^i \partial_i \rho = 0$, then we have by (5.13)

$$|\partial_\rho \partial_q \rho| \leq \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} (1 - D_{LL}/4 - E_{LL}/4)^{-1} \partial_r \rho \ln \left| \frac{1+t}{1+|\rho|} \right|. \tag{5.14}$$

In addition, by a direct computation, we have

$$\begin{aligned}
\partial_\rho \partial_q \rho &= \partial_\rho \left[\left(1 - \frac{1}{4} D_{LL} - \frac{1}{4} E_{LL} \right)^{-1} \partial_r \rho \right] \\
&= \left(1 - \frac{1}{4} D_{LL} - \frac{1}{4} E_{LL} \right)^{-1} \partial_\rho \partial_r \rho + \frac{1}{4} \left(1 - \frac{1}{4} D_{LL} - \frac{1}{4} E_{LL} \right)^{-2} \partial_\rho (D_{LL} + E_{LL}) \partial_r \rho.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
|\partial_\rho \partial_r \rho| &\leq \left| \left(1 - \frac{1}{4} D_{LL} - \frac{1}{4} E_{LL} \right) \partial_\rho \partial_q \rho \right| + \frac{1}{4} \left| \left(1 - \frac{1}{4} D_{LL} - \frac{1}{4} E_{LL} \right)^{-1} \partial_\rho (D_{LL} + E_{LL}) \partial_r \rho \right| \\
&\leq \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} (\partial_r \rho) \ln \left| \frac{1+t}{1+|\rho|} \right| + \frac{1}{2} |\rho_q^{-1} \partial_q (D_{LL} + E_{LL})| \partial_r \rho \\
&\leq \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} (\partial_r \rho) \ln \left| \frac{1+t}{1+|\rho|} \right| + \left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon} \frac{C \varepsilon}{(1+t)(1+|\rho|)^{\nu'}} \partial_r \rho.
\end{aligned} \tag{5.15}$$

Without loss of generality, $t > 2$ is assumed. When $(t, x) \in H$, $|\rho| \leq t/2$, and we further have

$\left(\frac{1+t}{1+|\rho|} \right)^{C\varepsilon} < \frac{1+t}{1+|\rho|}$ and $\ln \frac{1+t}{1+|\rho|} \geq \ln \frac{3}{2}$. This, together with (5.15), yields

$$|\partial_\rho \partial_r \rho| \leq \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} (\partial_r \rho) \ln \left| \frac{1+t}{1+|\rho|} \right| + \frac{C \varepsilon}{(1+|\rho|)^{1+\nu'}} \partial_r \rho$$

$$\begin{aligned}
&\leq \frac{C\varepsilon}{(1+|\rho|)^{1+\nu'}} \partial_r \rho \ln \left| \frac{1+t}{1+|\rho|} \right| \\
&\leq \frac{C\varepsilon \ln |1+t|}{(1+|\rho|)^{1+\nu'}} \partial_r \rho.
\end{aligned} \tag{5.16}$$

When $(t, x) \notin H$, $\rho = r - t$ and $\partial_\rho \partial_r \rho = 0$. Therefore, (5.1) of Lemma 5.1 holds.

Based on the preparations above, we now start to show Proposition 5.2. Using $\partial^{\mathbf{k}} Z^I u$ ($0 \leq \mathbf{k} \leq k$) instead of u in (4.5) yields

$$\begin{aligned}
&\int_{\Sigma_t} |\partial \partial^{\mathbf{k}} Z^I u|^2 W dx + \int_0^t \int_{\Sigma_\tau} (1+|q|)^{-3/2} |\bar{Z} \partial^{\mathbf{k}} Z^I u|^2 W dx d\tau \\
&\leq C \int_{\Sigma_0} |\partial \partial^{\mathbf{k}} Z^I u|^2 W dx + \int_0^t \int_{\Sigma_\tau} \frac{C\varepsilon}{1+\tau} |\partial \partial^{\mathbf{k}} Z^I u|^2 W dx d\tau \\
&\quad + \int_0^t \int_{\Sigma_\tau} \frac{C(1+\tau)}{\varepsilon} |\tilde{\square}_g \partial^{\mathbf{k}} Z^I u|^2 W dx d\tau.
\end{aligned} \tag{5.17}$$

Thus, we have

$$\begin{aligned}
&E_{k,i}(t) + \sum_{\mathbf{k} \leq k, I \leq i} \int_0^t \int_{\Sigma_\tau} (1+|q|)^{-3/2} |\bar{Z} \partial^{\mathbf{k}} Z^I u|^2 W dx d\tau \\
&\leq C E_{k,i}(0) + \int_0^t \frac{C\varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + \sum_{\mathbf{k} \leq k, I \leq i} \int_0^t \int_{\Sigma_\tau} \frac{C(1+\tau)}{\varepsilon} |\tilde{\square}_g \partial^{\mathbf{k}} Z^I u|^2 W dx d\tau.
\end{aligned} \tag{5.18}$$

Starting from (5.18), we now show (5.10). The proof procedure will be divided into the following three cases.

Case 1. $i = 0$

Without loss of generality, we may assume $1 \leq k \leq N$ since (5.10) obviously holds for $k = 0$. It follows from $\tilde{\square}_g u = 0$ and (3.34) that for $\mathbf{k} \leq k$

$$\tilde{\square}_g \partial^{\mathbf{k}} u = \sum_{s+n=\mathbf{k}, n < \mathbf{k}} C_{ij}^{lm} (\partial^s D^{ij}) \partial_l \partial_m \partial^n u + \sum_{s+n=\mathbf{k}, n < \mathbf{k}} G(\partial^s u, \partial^n u). \tag{5.19}$$

We note that the first term $\sum_{s+n=\mathbf{k}, n < \mathbf{k}} C_{ij}^{lm} (\partial^s D^{ij}) \partial_l \partial_m \partial^n u$ in (5.19) have been treated in Lemma 9.2 of [20] as follows

$$\left| \sum_{s+n=\mathbf{k}, n < \mathbf{k}} C_{ij}^{lm} (\partial^s D^{ij}) \partial_l \partial_m \partial^n u \right| \leq \frac{C\varepsilon}{1+t} \sum_{n=\mathbf{k}+1} |\partial^n u| + \frac{C\varepsilon}{(1+t)^{1-C\varepsilon}} \sum_{1 \leq n \leq \mathbf{k}} |\partial^n u|. \tag{5.20}$$

Therefore, we just need to treat the second term in (5.19). We rewrite it as

$$\sum_{s+n=\mathbf{k}, n < \mathbf{k}} G(\partial^s u, \partial^n u) = I_1 + I_2 \tag{5.21}$$

with

$$\begin{aligned} I_1 &= \sum_{n=\mathbf{k}-1} G(\partial u, \partial^n u) + G(\partial^{\mathbf{k}} u, u), \\ I_2 &= \sum_{s+n=\mathbf{k}, 2 \leq s \leq \mathbf{k}-1, 1 \leq n \leq \mathbf{k}-2} G(\partial^s u, \partial^n u). \end{aligned}$$

By Lemma 2.5, one has

$$I_1 \leq C(|\bar{Z}\partial u| \sum_{n=\mathbf{k}+1} |\partial^n u| + |\partial^2 u| |\bar{Z}\partial^{\mathbf{k}} u|). \quad (5.22)$$

For the first term on the right hand side of (5.22), we use $|\bar{Z}\partial u| \leq C(1+t)^{-1}|Z\partial u|$. And for the second term we use $|\partial^2 u| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-1-\nu'}$. Then by (5.9) and $\nu' > 1/2$ we obtain

$$\begin{aligned} |I_1| &\leq C(1+t)^{-1}|Z\partial u| \sum_{n=\mathbf{k}+1} |\partial^n u| + C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-1-\nu'} |\bar{Z}\partial^{\mathbf{k}} u| \\ &\leq C(1+t)^{-1} \frac{C\varepsilon}{1+t} (1+t)^{C\varepsilon} \sum_{n=\mathbf{k}+1} |\partial^n u| + C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-3/2} |\bar{Z}\partial^{\mathbf{k}} u|. \end{aligned} \quad (5.23)$$

In addition, by (5.9) for $s < N'$ or $n+1 < N'$, we have

$$|I_2| \leq C \sum_{m+l=\mathbf{k}+3, 2 \leq m \leq \mathbf{k}, 1 \leq l \leq \mathbf{k}} |\partial^m u| |\partial^l u| \leq \frac{C\varepsilon}{(1+t)^{1-C\varepsilon}} \sum_{1 \leq n \leq \mathbf{k}} |\partial^n u|. \quad (5.24)$$

Then it follows (5.23)-(5.24) that

$$\begin{aligned} &\sum_{\mathbf{k} \leq k} \int_0^t \int_{\Sigma_\tau} \frac{(1+\tau)}{\varepsilon} \left| \sum_{s+n=\mathbf{k}, n < \mathbf{k}} G(\partial^s u, \partial^n u) \right|^2 W dx d\tau \\ &\leq \int_0^t \frac{C\varepsilon}{1+\tau} E_{k,0}(\tau) d\tau + \int_0^t \frac{C\varepsilon}{(1+\tau)^{1-C\varepsilon}} E_{k-1,0}(\tau) d\tau \\ &\quad + \sum_{\mathbf{k} \leq k} \int_0^t \int_{\Sigma_\tau} C\varepsilon(1+t)^{-1+C\varepsilon}(1+|q|)^{-3/2} |\bar{Z}\partial^{\mathbf{k}} u|^2 W dx d\tau. \end{aligned} \quad (5.25)$$

Combining (5.25) and (5.18) yields

$$\begin{aligned} E_{k,0}(t) &< E_{k,0}(t) + \sum_{\mathbf{k} \leq k} \int_0^t \int_{\Sigma_\tau} [1 - C\varepsilon(1+t)^{-1+C\varepsilon}](1+|q|)^{-3/2} |\bar{Z}\partial^{\mathbf{k}} u|^2 W dx d\tau \\ &\leq C E_{k,0}(0) + \int_0^t \frac{C\varepsilon}{1+\tau} E_{k,0}(\tau) d\tau + \int_0^t \frac{C\varepsilon}{(1+\tau)^{1-C\varepsilon}} E_{k-1,0}(\tau) d\tau, \end{aligned} \quad (5.26)$$

which completes the proof of (5.10) for $i = 0$.

Case 2. $k = 0$

By (3.33) and $\tilde{\square}_g u = 0$, we have

$$\tilde{\square}_g Z^I u = \sum_{J+K \leq I, K < I} C_{JKij}^{I \quad lm} (Z^J D^{ij}) \partial_l \partial_m Z^K u + \sum_{J+K \leq I, K < I} G(Z^J u, Z^K u). \quad (5.27)$$

For the first term $\sum_{J+K \leq I, K < I} C_{JKij}^{I \quad lm} (Z^J D^{ij}) \partial_l \partial_m Z^K u$ in (5.27) has been treated in Lemma 9.2 of [20]) as follows

$$\begin{aligned} & \frac{1}{\varepsilon} \sum_{I \leq i} \int_0^t (1+\tau) \int \left| \sum_{J+K \leq I, K < I} C_{JKij}^{I \quad lm} (Z^J D^{ij}) \partial_l \partial_m Z^K u \right|^2 W dx \\ & \leq \int_0^t \frac{C\varepsilon}{1+\tau} E_{0,i}(\tau) d\tau + \int_0^t \frac{C\varepsilon}{(1+\tau)^{1-C\varepsilon}} E_{1,i-1}(\tau) d\tau. \end{aligned} \quad (5.28)$$

Hence we only deal with the second term $\sum_{J+K \leq I, K < I} G(Z^J u, Z^K u)$ in (5.27). Note that

$$\begin{aligned} \sum_{J+K \leq I, K < I} G(Z^J u, Z^K u) &= G(Z^I u, u) + \sum_{J \leq 1, K = I-1} G(Z^J u, Z^K u) \\ &+ \sum_{J \leq I-1, K \leq I-2} G(Z^J u, Z^K u). \end{aligned} \quad (5.29)$$

For term $G(Z^I u, u)$, as treated for I_1 in (5.22), we have

$$\begin{aligned} |G(Z^I u, u)| &\leq C (|\bar{Z} Z^I u| |\partial^2 u| + |\partial Z^I u| |\bar{Z} \partial u|) \\ &\leq C\varepsilon (1+t)^{-1+C\varepsilon} (1+|q|)^{-3/2} |\bar{Z} Z^I u| + \frac{C\varepsilon}{(1+t)^2} (1+t)^{C\varepsilon} |\partial Z^I u|. \end{aligned}$$

For the left terms $\sum_{J \leq 1, K = I-1} G(Z^J u, Z^K u)$ and $\sum_{J \leq I-1, K \leq I-2} G(Z^J u, Z^K u)$ in (5.29), their estimates follow directly from (5.9).

Then proceeding as in Case 1, we finally arrive at

$$E_{0,i}(t) \leq C E_{0,i}(0) + \int_0^t \frac{C\varepsilon}{1+\tau} E_{0,i}(\tau) d\tau + \int_0^t \frac{C\varepsilon}{(1+\tau)^{1-C\varepsilon}} E_{1,i-1}(\tau) d\tau.$$

Case 3. $k \geq 1, i \geq 1$

The case proceeds similarly as in Case 1 and 2.

Combining all the three cases above, we complete the proof of Proposition 5.2. \square

Based on Proposition 5.2, as in the proof of Proposition 9.1 of [20], we have

Proposition 5.3. *Assume that the assumptions in Proposition 5.2 are valid. Then for $0 \leq t < T$,*

$$E_{k,i}(t) \leq C \sum_{l=0}^i E_{k+l,i-l}(0)(1+t)^{C\varepsilon}, \quad k+i \leq N. \quad (5.30)$$

where $C > 0$ is independent of T .

Proof. Since we have established the crucial Proposition 5.2, the proof of (5.30) is completely similar to that in Proposition 9.1 of [20] by Gronwall's inequality, we omit the proof here. \square

§6. The proof of Theorem 1.1.

In this section, we complete the proofs of the weak decay estimate (3.1) and Theorem 1.1 by continuity method. Let $N \geq 8$ be fixed and set

$$E_N(t) = \sum_{I \leq N} \int |\partial Z^I u(t, x)|^2 dx.$$

Without loss of generality, we assume

$$E_N(0) \leq \varepsilon^2$$

and also assume for fixed $0 < \delta < \frac{1}{2}$ and $0 \leq t \leq T$,

$$E_N(t) \leq C\varepsilon^2(1+t)^\delta. \quad (6.1)$$

Next, we derive the weak decay estimate (3.1). First it follows from (3.33) and $\tilde{\square}_g u = 0$ that

$$\begin{aligned} |\square Z^I u| &= |(\tilde{\square}_g - D^{ij} \partial_i \partial_j - E^{ij} \partial_i \partial_j) Z^I u| \\ &\leq |\tilde{\square}_g Z^I u| + C|u| \partial^2 Z^I u + |G(u, Z^I u)| \\ &\leq C \sum_{J+K \leq I} |Z^J u| |\partial^2 Z^K u| + \sum_{J+K \leq I} |G(Z^J u, Z^K u)| \\ &\leq \frac{C}{1+|q|} \sum_{J+K \leq I+1} |Z^J u| |\partial Z^K u| + C \sum_{J+K \leq I} |\partial Z^J u| |\partial^2 Z^K u|. \end{aligned} \quad (6.2)$$

Then by Hölder inequality and $\|(1+|q|)^{-1} Z^J u\|_{L^2} \leq \|\partial Z^J u\|_{L^2}$, we have for $I \leq N-3$

$$\begin{aligned} &\sum_{L \leq 2} \int |Z^L \square Z^I u(t, x)| dx \\ &\leq C \sum_{L \leq I+2} \int |\square Z^L u(t, x)| dx \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{J+K \leq I+3} \int \frac{1}{1+|q|} |Z^J u| |\partial Z^K u| dx + C \sum_{J+K \leq I+2} \int |\partial Z^J u| |\partial^2 Z^K u| dx \\
&\leq C \sum_{J+K \leq I+3} \|\partial Z^J u\|_{L^2} \|\partial Z^K u\|_{L^2} \\
&\leq C E_N(t).
\end{aligned} \tag{6.3}$$

In addition, by Corollary 10.3 of [20], for $\phi(0, x) = \partial_t \phi(0, x) = 0$ when $|x| \geq 1$, then

$$|\phi(t, x)|(1+t+|x|) \leq C \sum_{I \leq 2} \int_0^t \int \frac{|(Z^I \square \phi)(\tau, y)|}{1+\tau+|y|} dy d\tau + C \sum_{I \leq 2} \|\partial Z^I \phi(0, \cdot)\|_{L^2}. \tag{6.4}$$

Therefore, it follows from (6.3)-(6.4) together with (6.1) that

$$\begin{aligned}
|Z^I u(t, x)|(1+t+r) &\leq C \int_0^t \frac{E_N(\tau)}{1+\tau} d\tau + C E_N(0) \\
&\leq C \int_0^t \frac{\varepsilon^2(1+\tau)^\delta}{1+\tau} d\tau + C \varepsilon^2 \\
&\leq C \varepsilon^2 \delta^{-1} (1+t)^\delta \\
&\leq C \varepsilon (1+t)^\delta.
\end{aligned}$$

Hence, we obtain (3.1) with $\nu = 1 - \delta > 1/2$, then the weak decay (3.1) holds. Thus, the estimates in Proposition 3.9 and Lemma 2.2- Lemma 2.4 are also true. It then follows that all the assumptions of Proposition 5.2 hold. Then by (5.30) and $W \geq 1$, we have

$$E_N(t) \leq E_{0,N}(t) \leq C \sum_{l=0}^N E_{l,N-l}(0) (1+t)^{C\varepsilon} \leq C \varepsilon^2 (1+t)^{C\varepsilon},$$

here we use the fact of $E_{k,i}(0) \leq C E_{k+i}(0)$ due to $0 < \varphi(q) = \int_{-\infty}^q \varphi'(s) ds = \int_{-\infty}^q (1+|s|)^{-3/2} ds \leq C$. This yields the proof the Theorem 1.1 by the continuity method and the local existence of the solution to (1.3) (In fact, only by a rough estimate as in [4, Theorem 1], one can then derive that the C^∞ -solution u of (1.3) exists for $t \in [0, T]$ with $T \geq e^{\frac{C}{\varepsilon}}$). In addition, if we choose $\delta = C\varepsilon$ for some suitable positive $C > 0$, as is seen from the proofs above, we can derive (3.1) with $\nu = 1 - C\varepsilon$, which means that the solution u to (1.3) does not behave like a solution to the 3-D free wave equation.

Acknowledgements. *Liu Yingbo and Yin Huicheng wish to express their gratitude to Professor Witt Ingo, University of Göttingen for many fruitful discussions in this problem when they were visiting the Mathematical Institute of the University of Göttingen from February to March of 2013. Ding Bingbing is also thankful to Professor Witt Ingo for his much guidance and huge helps when she read PhD in University of Göttingen from October of 2012 to February of 2014 under the supervision of Professor Witt Ingo.*

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